Question:

Let: $x + y + z = 1$
$x^2 + y^2 + z^2 = 4$
$x^3 + y^3 + z^3 = 9$

Solve for $x^4 + y^4 + z^4$

For extra credit, solve for the general case where:
$a = x + y + z$
$b = x^2 + y^2 + z^2$
$c = x^3 + y^3 + z^3$

Solution:

Let's jump right in and start with the general "extra credit" case.

Let $d = x^4 + y^4 + z^4$

We can get to d with $a^4 = (x+y+z)^4 = x^4 + y^4 + z^4 + 4*(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) + 6*(x^2y^2 + x^2z^2 + y^2z^2) + 12*(x^2yz + xy^2z + xyz^2)$

You can see there are lots of mixed terms in that expression we will have to evaluate.

Let's start at the bottom and work our way up. Evaluating a^2 is a good place to start.

$a^2 = (x + y + z)^2 =$
$x^2 + y^2 + z^2 + 2*(xy + xz + yz) =$
$b + 2(xy + xz + yz)$ -- Recall: $b = x^2 + y^2 + z^2$
(1) $xy + xz + yz = (a^2 - b)/2$

$ab = (x + y + z)*(x^2 + y^2 + z^2)$
$= x^3 + y^3 + z^3 + xy^2 + xz^2 + yx^2 + yz^2 + zx^2 + zy^2$
$= c + xy^2 + xz^2 + yx^2 + yz^2 + zx^2 + zy^2$ -- Recall: $c = x^3 + y^3 + z^3$
(2) $xy^2 + xz^2 + yx^2 + yz^2 + zx^2 + zy^2 = ab - c$

$a^3 = (x + y + z)^3 =$
$x^3 + y^3 + z^3 + 3(xy^2 + xz^2 + yx^2 + yz^2 + zx^2 + zy^2) + 6xyz =$
$c + 3(ab - c) + 6xyz$
$6xyz = a^3 - c - 3(ab-c)$

$6xyz = a^3 + 2c - 3*ab$
(3) $xyz = (a^3 + 2c - 3ab)/6$

$b^2 = (x^2 + y^2 + z^2)^2 = x^4 + y^4 + z^4 + 2*(x^2y^2 + x^2z^2 + y^2z^2)$
(4) $x^2y^2 + x^2z^2 + y^2z^2 = (b^2 - d)/2$ -- Recall: $d = x^4 + y^4 + z^4$

$a^2 * b = (x^2 + y^2 + z^2 + 2*(xy + xz + yz)) * (x^2 + y^2 + z^2) =$
$x^4 + y^4 + z^4 + 2*(x^2y^2 + x^2z^2 + y^2z^2) + 2(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) + 2(x^2yz + xy^2z + xyz^2) =$
$d + 2*(x^2y^2 + x^2z^2 + y^2z^2) + 2(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) + 2xyz*(x+y+z) =$
$d + 2*(x^2y^2 + x^2z^2 + y^2z^2) + 2(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) + a(a^3 + 2c - 3ab)/3 =$
$d + (b^2 - d) + 2(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) + a(a^3 - 3ab + 2c)/3 =$
$b^2 + 2(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) + a(a^3 - 3ab+ 2c)/3$

$2(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) = a^2 * b - b^2 - a(a^3 - 3ab+ 2c)/3$
$x^3y + x^3z + y^3x + y^3z + z^3x + z^3y = a^2 * b/2 - b^2/2 - a^4/6 + a^2b/2 - ac/3$
(5) $x^3y + x^3z + y^3x + y^3z + z^3x + z^3y = a^2 * b - b^2/2 - a^4/6 - ac/3$

$a^4 = (x + y + z)^4 =$
$x^4 + y^4 + z^4 + 4*(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) + 6*(x^2y^2 +x^2z^2 + y^2z^2) + 12*(x^2yz + xy^2z + xyz^2) =$
$d + 4*(a^2b - b^2/2 - a^4/6 - ac/3) + 6((b^2 - d)/2) + 12xyz(x+y+z) =$
$d + 4a^2b - 2b^2 - (2/3)a^4 - (4/3)ac + 3b^2 - 3d + 12((a^3 + 2c - 3ab)/6)*a =$
$d + 4a^2b - 2b^2 - (2/3)a^4 - (4/3)ac + 3b^2 - 3d + 2a^4 + 4ac - 6a^2b =$
$4/3a^4 + (8/3)ac - 2a^2b + b^2$

$2d = 1/3a^4 + (8/3)ac - 2a^2b$
$d = a^4/6 + (4/3)ac - a^2b + b^2/2$

Going back to the original problem, where a = 1, b = 4, and c = 9:

$d = 1^4/6 + (4/3)1*9 - 1^2*4 + 4^2/2 = 97/6 = 16.166666.$

Further reading:

Newton identities: https://en.wikipedia.org/wiki/Newton's_identities