Question:

Let:  \( x + y + z = 1 \)
\( x^2 + y^2 + z^2 = 4 \)
\( x^3 + y^3 + z^3 = 9 \)

Solve for \( x^4 + y^4 + z^4 \)

For extra credit, solve for the general case where:
\( a = x + y + z \)
\( b = x^2 + y^2 + z^2 \)
\( c = x^3 + y^3 + z^3 \)

Solution:

Let's jump right in and start with the general "extra credit" case.

Let \( d = x^4 + y^4 + z^4 \)

We can get to \( d \) with
\[
\begin{align*}
\left( x + y + z \right)^4 &= x^4 + y^4 + z^4 + 4 \left( x^3 y + x^3 z + y^3 x + y^3 z + z^3 x + z^3 y \right) + \\
&\quad 6 \left( x^2 y^2 + x^2 z^2 + y^2 z^2 \right) + 12 \left( x^2 yz + xy^2 z + xyz^2 \right)
\end{align*}
\]

You can see there are lots of mixed terms in that expression we will have to evaluate.

Let's start at the bottom and work our way up. Evaluating \( a^2 \) is a good place to start.

\[
\begin{align*}
a^2 &= \left( x + y + z \right)^2 = \\
&= x^2 + y^2 + z^2 + 2 \left( xy + xz + yz \right) = \\
&= b + 2 \left( xy + xz + yz \right) \quad \text{-- Recall: } b = x^2 + y^2 + z^2 \\
(1) \quad xy + xz + yz &= \left( a^2 - b \right)/2 \\

ab &= \left( x + y + z \right) \cdot \left( x^2 + y^2 + z^2 \right) \\
&= x^3 + y^3 + z^3 + xy^2 + xz^2 + yx^2 + yz^2 + zx^2 + zy^2 \\
&= c + xy^2 + xz^2 + yx^2 + yz^2 + zx^2 + zy^2 \quad \text{-- Recall: } c = x^3 + y^3 + z^3 \\
(2) \quad xy^2 + xz^2 + yx^2 + yz^2 + zx^2 + zy^2 &= ab - c \\

a^3 &= \left( x + y + z \right)^3 = \\
&= x^3 + y^3 + z^3 + 3 \left( xy^2 + xz^2 + yx^2 + yz^2 + zx^2 + zy^2 \right) + 6xyz = \\
&= c + 3 \left( ab - c \right) + 6xyz \\
6xyz &= a^3 - c - 3 \left( ab - c \right) 
\end{align*}
\]
\[6xyz = a^2 + 2c - 3ab\]
\[(3) \text{ xyz } = (a^3 + 2c - 3ab)/6\]
\[b^2 = (x^2 + y^2 + z^2)^2 = x^4 + y^4 + z^4 + 2(x^2y^2 + x^2z^2 + y^2z^2)\]
\[(4) x^2y^2 + x^2z^2 + y^2z^2 = (b^2 - d)/2 \quad \text{-- Recall: } \text{ d } = x^4 + y^4 + z^4\]
\[a^2 * b = (x^2 + y^2 + z^2 + 2(xy + xz + yz)) * (x^2 + y^2 + z^2) = x^4 + y^4 + z^4 + 2(x^2y^2 + x^2z^2 + y^2z^2) + 2(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) + 2(x^2yz + xy^2z + xz^2y)\]
\[d + 2(x^2y^2 + x^2z^2 + y^2z^2) + 2(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) + 2xyz(x+y+z) = d + 2(x^2y^2 + x^2z^2 + y^2z^2) + 2(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) + a(a^3 + 2c - 3ab)/3 = d + (b^2 - d) + 2(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) + a(a^3 - 3ab + 2c)/3 = b^2 + 2(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) + a(a^3 - 3ab + 2c)/3\]
\[2(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) = a^2 * b - b^2 - a(a^3 - 3ab + 2c)/3\]
\[x^3y + x^3z + y^3x + y^3z + z^3x + z^3y = a^2 * b/2 - b^2/2 - a^4/6 + a^2b/2 - ac/3\]
\[(5) x^3y + x^3z + y^3x + y^3z + z^3x + z^3y = a^2 * b - b^2/2 - a^4/6 - ac/3\]
\[a^4 = (x + y + z)^4 = x^4 + y^4 + z^4 + 4(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) + 6*(x^2y^2 + x^2z^2 + y^2z^2) + 12*(x^2yz + xy^2z + xz^2y) = d + 4*(a^2b - b^2/2 - a^4/6 - ac/3) + 6((b^2 - d)/2) + 12xyz(x+y+z) = d + 4a^2b - 2b^2 - (2/3)a^4 - (4/3)ac + 3b^2 - 3d + 12((a^3 + 2c - 3ab)/6)*a = d + 4a^2b - 2b^2 - (2/3)a^4 - (4/3)ac + 3b^2 - 3d + 2a^4 + 4ac - 6a^2b = 4/3a^4 + (8/3)ac - 2a^2b + b^2\]
\[2d = 1/3a^4 + (8/3)ac - 2a^2b\]
\[d = a^4/6 + (4/3)ac - a^2b + b^2/2\]

Going back to the original problem, where \(a = 1\), \(b = 4\), and \(c = 9\):
\[d = 1^4/6 + (4/3)1*9 - 1^2*4 + 4^2/2 = 97/6 = 16.166666.\]

Further reading: