Q: You want to get from the bank to the casino by bus. There are two buses that make a loop stopping at both places continuously. It takes one bus ½ hour to complete a loop and the other bus ¾ hour. You have no idea where they are in their loops. What is the average waiting time until the next bus?

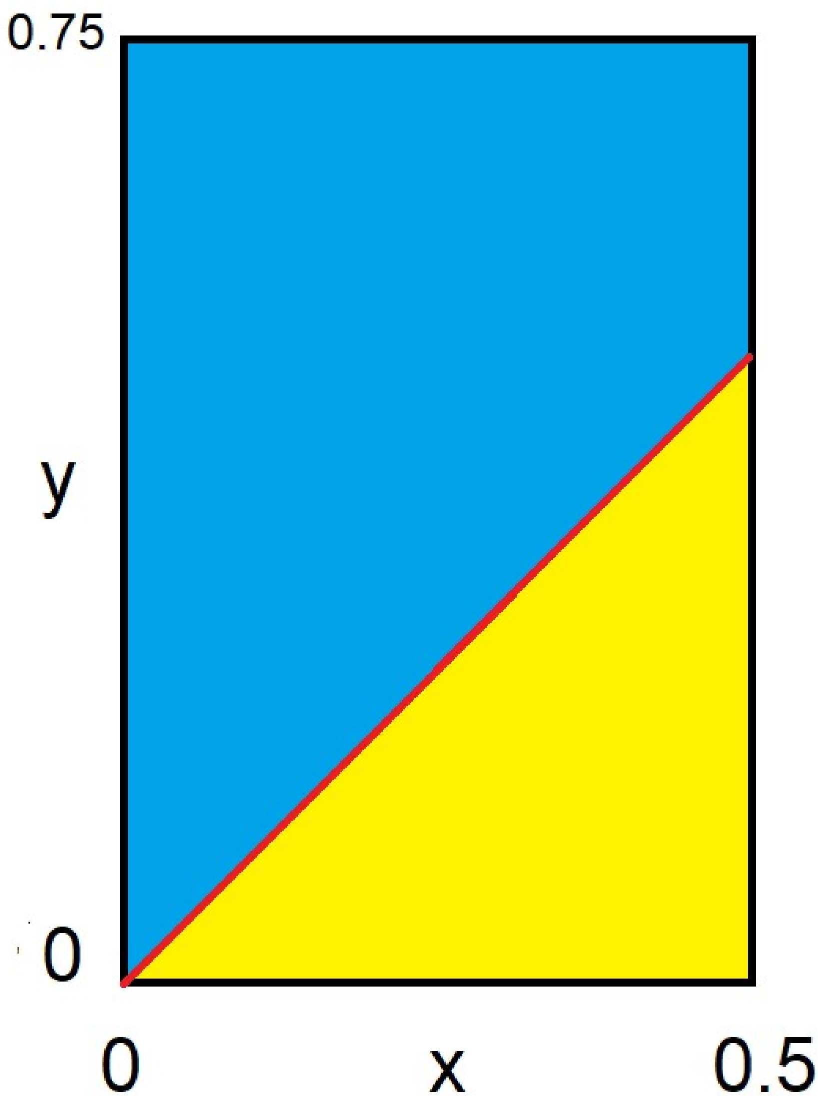
A: 7/36 hour = 11 2/3 minutes = 11 minutes, 40 seconds.

Solution:

Let x = time until 30-minute bus arrives.

Let y = time until 45-minute bus arrives.

Consider the following diagram showing all possible combinations of x and y. The yellow region represents those combinations of x and y where bus y arrives first. The blue region represents those combinations where bus x arrives first. Note that the blue region is larger, because x makes a loop in less time, and thus comes by the bank more often.



First, let's calculate the total wait time if x<y. In other words, the yellow region. That can be expressed as:

$$\int_0^{0.5}\int_0^{x} x\,dy\,dx =$$

$$\int_0^{0.5} xy \text{ from 0 to x } dy =$$

$$\int_0^{0.5} x^2\ dx =$$

$$\frac{x^3}{3} \text{ from 0.5 to 0} =$$

(1/8)/3 = 1/24

Second, let's calculate the total wait time if x>y. In other words, the blue region. That can be expressed as:

$$\int_0^{0.5}\int_x^{0.75} x\,dy\,dx =$$

$$\int_0^{0.5} xy \text{ from x to } 0.75\ dx =$$

$$\int_0^{0.5} x(0.75 - x)\,dx =$$

$$\int_0^{0.5} 0.75x - x^2\,dx =$$

$\frac{3x^2}{8}$ - $\frac{x^3}{3}$ from 0 to 0.5 =

$\frac{3}{32}$ - $\frac{1}{24}$ = $\frac{5}{96}$

Next, let's add the yellow and blue areas:

$\frac{1}{48} + \frac{5}{96} = \frac{7}{96}$

Next, let's divide by the total area of 3/8 to get an average wait time:

$\frac{7}{96}$ / $\frac{3}{8}$ = $\frac{7}{36}$ hours

7/36 hours = 11 2/3 minutes = 11 minutes, 40 seconds.