Question

A sign has ten light bulb sockets, each with a light bulb in it. Each socket takes a different size light bulb. Besides the light bulb already in each socket, there is one spare per socket. The life of each light bulb is distributed exponentially*, with a mean lifetime of one day. As soon as a light bulb dies, the spare will immediately replace it, if there still is a spare for that socket.

How long until the last light bulb burns out?

* An exponential distribution means the light bulb has a memoryless property, where it is never any closer to burning out, regardless how long it has been on. For short periods of time, the probability of burning out can be very closely approximated as the ratio of that period of time to the expected lifetime.

Answer

Approximately 4.622957 days.

Brute Force Solution

There are 55 possible states, between the number of original light bulbs burning and number of replacement light bulbs burning. From any given state, either an original light bulb can burn out or a replacement (assuming at least one replacement is burning).

Table 1 shows the number of light bulbs burning along the left column and number of them that are originals along the top row. The body of the table shows the probability that the bulbs will ever be in that state. For example, the probability there will at some point be 5 light bulbs left, with 3 of them being originals is 30.619%.

Table 1

Burning	10	9	8	7	6	5	4	3	2	1	0
10	1.00000	1.00000	0.90000	0.72000	0.50400	0.30240	0.15120	0.06048	0.01814	0.00363	0.00036
9		0.10000	0.28000	0.46489	0.56318	0.52665	0.38331	0.21269	0.08541	0.02225	0.00283
8			0.03111	0.13442	0.30534	0.46308	0.50237	0.39298	0.21380	0.07322	0.01199
7				0.01680	0.09314	0.25349	0.43225	0.49261	0.37147	0.17021	0.03630
6					0.01331	0.08573	0.25669	0.45262	0.49164	0.30977	0.08793
5						0.01429	0.09985	0.30619	0.51148	0.46273	0.18048
4							0.01997	0.14245	0.41372	0.57705	0.32474
3								0.03561	0.24247	0.59443	0.52289
2									0.08082	0.47711	0.76144
1										0.23856	1.00000

Recall that if there are n light bulbs, each with a mean lifetime of x, then the mean lifetime of the next failure is x/n. That said, table 2 shows the expected time the light bulbs will be in any possible state.

Table 2

Burning	10	9	8	7	6	5	4	3	2	1	0
10	0.10000	0.10000	0.10000	0.10000	0.10000	0.10000	0.10000	0.10000	0.10000	0.10000	0.10000
9		0.11111	0.11111	0.11111	0.11111	0.11111	0.11111	0.11111	0.11111	0.11111	0.11111
8			0.12500	0.12500	0.12500	0.12500	0.12500	0.12500	0.12500	0.12500	0.12500
7				0.14286	0.14286	0.14286	0.14286	0.14286	0.14286	0.14286	0.14286
6					0.16667	0.16667	0.16667	0.16667	0.16667	0.16667	0.16667
5						0.20000	0.20000	0.20000	0.20000	0.20000	0.20000
4							0.25000	0.25000	0.25000	0.25000	0.25000
3								0.33333	0.33333	0.33333	0.33333
2									0.50000	0.50000	0.50000
1										1.00000	1.00000

Table 3 shows the expected time in each state. Each cell is the product of the tables in that position of tables 1 and 2. For example, the time in which there are five bulbs left, including three originals, is 0.06124 days.

Table 3

Burning	10	9	8	7	6	5	4	3	2	1	0
10	0.10000	0.10000	0.09000	0.07200	0.05040	0.03024	0.01512	0.00605	0.00181	0.00036	0.00004
9	0.00000	0.01111	0.03111	0.05165	0.06258	0.05852	0.04259	0.02363	0.00949	0.00247	0.00031
8	0.00000	0.00000	0.00389	0.01680	0.03817	0.05788	0.06280	0.04912	0.02673	0.00915	0.00150
7	0.00000	0.00000	0.00000	0.00240	0.01331	0.03621	0.06175	0.07037	0.05307	0.02432	0.00519
6	0.00000	0.00000	0.00000	0.00000	0.00222	0.01429	0.04278	0.07544	0.08194	0.05163	0.01466
5	0.00000	0.00000	0.00000	0.00000	0.00000	0.00286	0.01997	0.06124	0.10230	0.09255	0.03610
4	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00499	0.03561	0.10343	0.14426	0.08119
3	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.01187	0.08082	0.19814	0.17430
2	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.04041	0.23856	0.38072
1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.23856	1.00000

The answer is the sum of the expected time in all possible states. In other words, the sum of the body of table 3. This sum is approximately 4.622957 days.

Calculus Solution

After x days, the probability any given light bulb is still burning is e^{-x}.

After x days, the probability of exactly one light bulb failure is xe^{-x}.

Assume an infinite supply of light bulbs for every socket, the probability of two or more failures per socket after x days is (1 - e^{-x} - xe^{-x}).

After x days, the probability of two or more failures in all ten sockets is $(1 - e^{-x} - xe^{-x})^{10}$

After x days, the probability of anything else (at least one socket is still on the first or second bulb) is 1 - $(1 - e^{-x} - xe^{-x})^{10}$.

The expected time at least one socket is still on it's first or second bulb can be expressed as.

$$\int_0^{\infty} 1 - (1 - e^{-x} - xe^{-x})^{10}$$

At this point, I suggest using an integral calculator. I like the one at https://www.integral-calculator.com/. Putting that in gives us an answer of:

335641897646511216668163083 / 726032911411261440000000000 = apx. 4.622957063944816.