You are given:

There is an airplane eight miles directly above a surface to air missile, which is fired at that moment.

At all times, the airplane travels in a straight direction.

The airplane travels at 600 miles per hour.

The missile travels at 2000 miles per hour.

The missile always travels at an angle that directly faces the airplane.

Image source: www.airvectors.net/avnsam.html

Questions:

1. How far will the plane travel before struck by the missile?
2. How long will it take for the missile to strike the plane?
3. How long is the flight path of the missile?

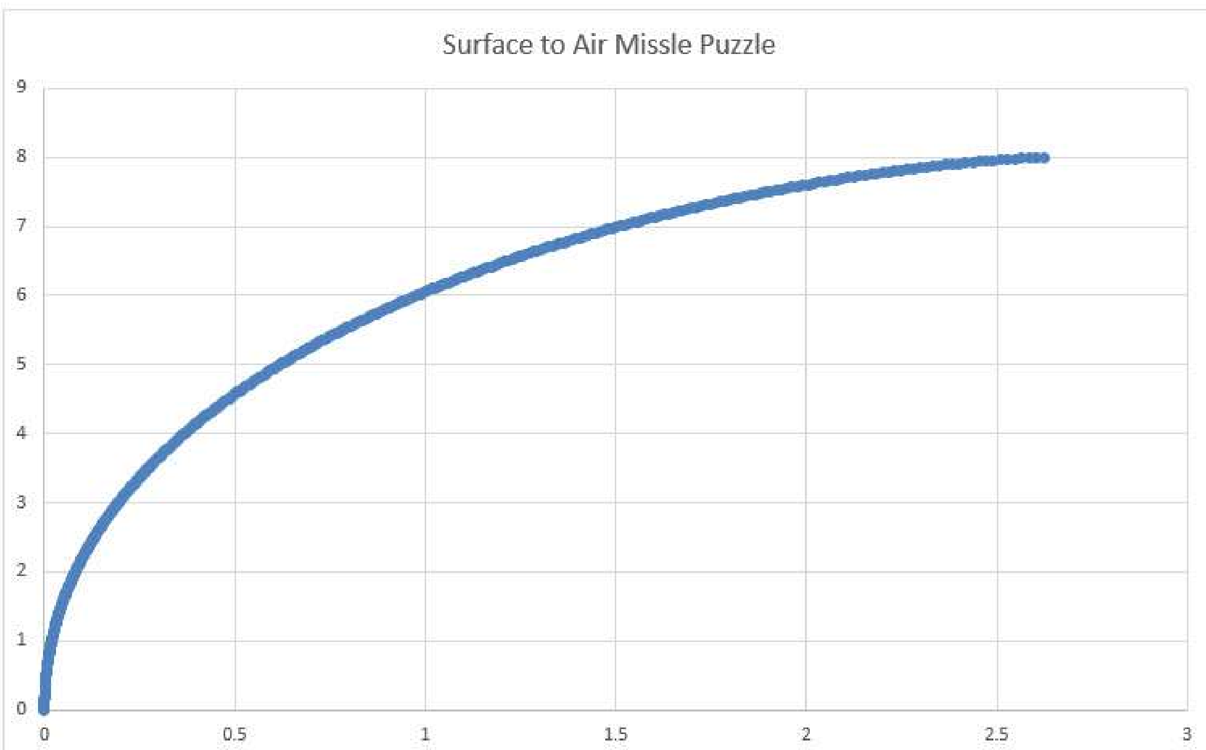
Answers:

How far will the plane travel before struck by the missile? = 240/91 miles

How long will it take for the missile to strike the plane? = 2/455 hours

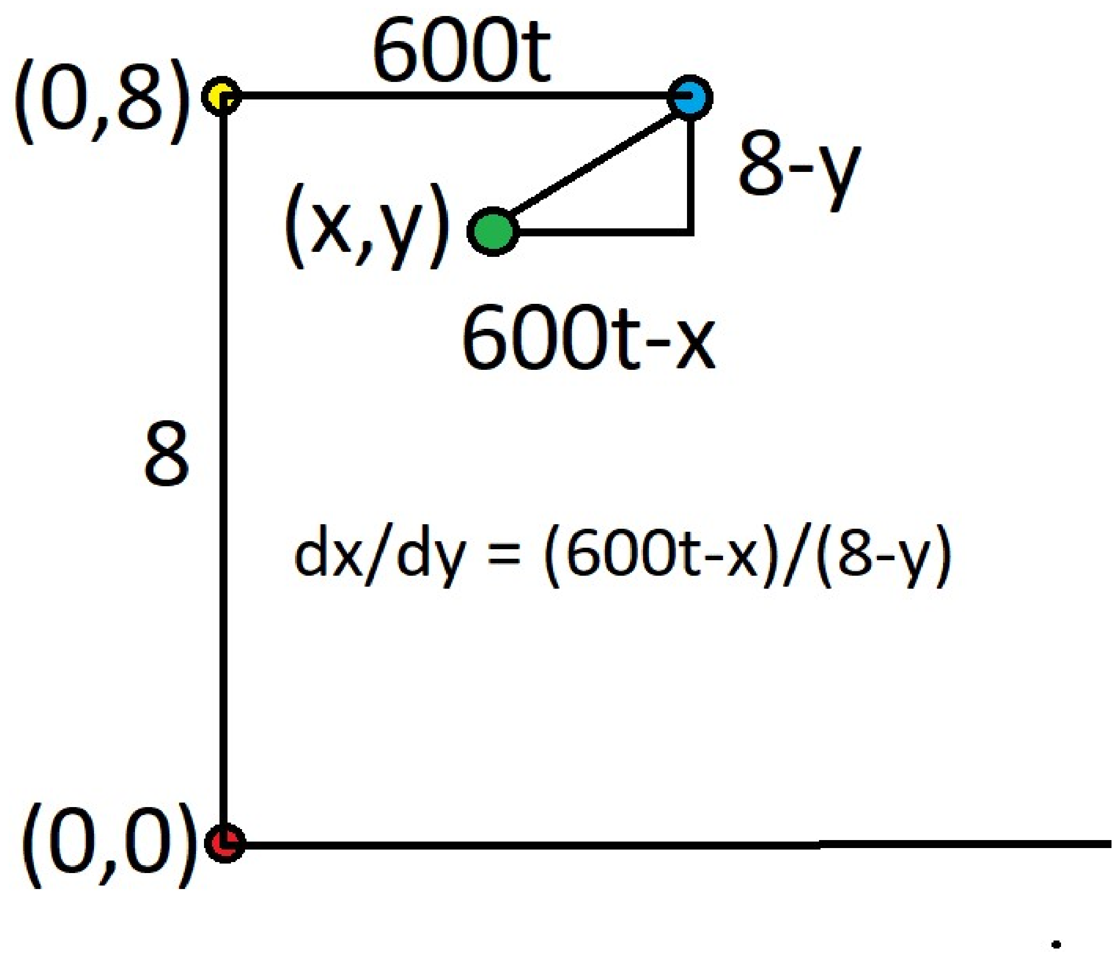
How long will the missile travel? = 800/91 miles

The following image shows the path of the missile.



Solution:

Consider the following diagram.



The red circle is the location of the missile launcher at (0,0)

The yellow circle is the location of the plane when the missile is launched at (0,8)

The blue circle is the location of the plane t hours later at (600t,8)

The green circle is the location of the missile at (x,y)

The hypotenuse of the triangle shows the direction of the missile at that moment.

The angle the missile is pointing at is $dx/dy = (600t-x)/(8-y)$.

Multiply both sides by (8-y):

$(8-y)\ dx/dy = 600t-x$

Take the derivative of both sides with respect to y:

$(8-y)\ dx^2/d^2y\ - dx/dy = 600\ dt/dy - dx/dy$

Add dx/dy to each side:

$(8-y)\ dx^2/d^2y = 600\ dt/dy$

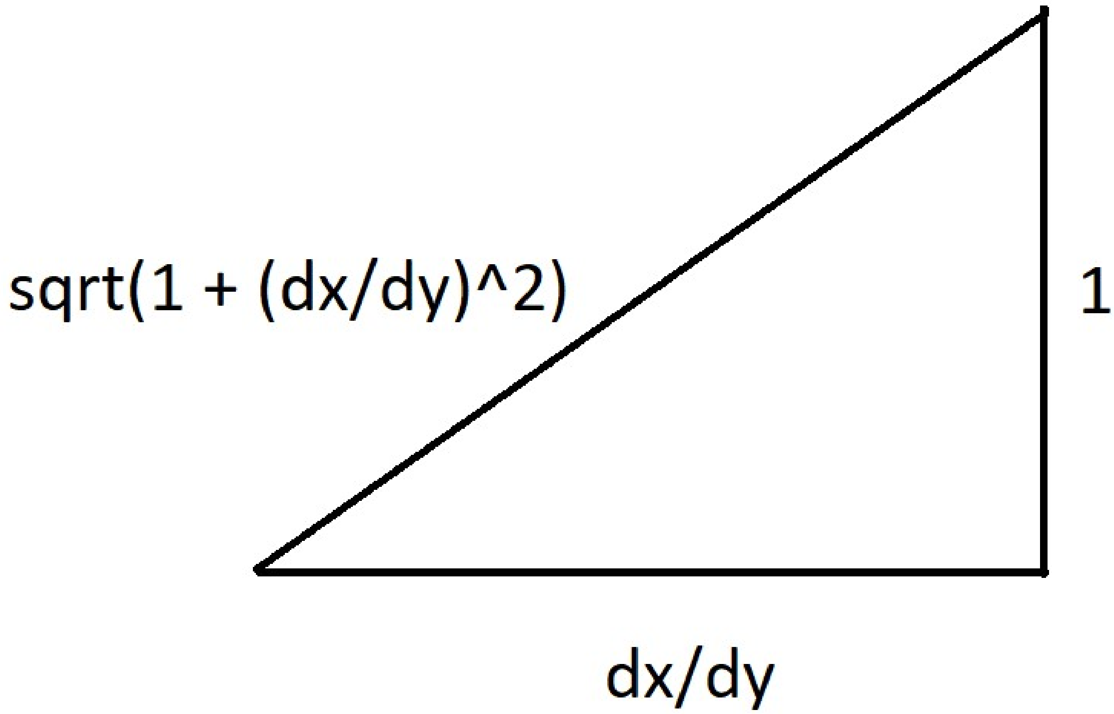
Let $u = dx/dy$

Equation 1: $(8-y)\ du/dy = 600\ dt/dy$

Let $dt/dy = (dt/ds) \times (ds/dy)$, where s = distance traveled by missile

We're given $ds/dt = 2000$, so $dt/ds = 1/2000$.

Let's look at the angle of attack of the missile, setting the y distance as 1.



Let s be the distance traveled by the missile. Then ds/dy = $\sqrt{1+(\frac{dx}{dy})^2}$

Resuming from equation 1:

(8-y) du/dy = 600 (1/2000 × $\sqrt{1+(\frac{dx}{dy})^2}$)

(8-y) du/dy = 0.3 × $\sqrt{1+(\frac{dx}{dy})^2}$

Let's move the dy to the other side:

(8-y) du= 0.3 × $\sqrt{1+(\frac{dx}{dy})^2}$ dy

Recall that u = dx/dy, so:

(8-y) du= 0.3 × $\sqrt{1+u^2}$ dy

Let's get the y and u terms together:

$\frac{1}{\sqrt{1+u^2}}$ du = 0.3 $\frac{1}{(8-y)}$ dy

Next, integrate both sides. I will prove the integral of $\frac{1}{\sqrt{1+u^2}}$ later, if you don't want to take it on faith.

ln | $\sqrt{1+\ u^2}$ + u | = -0.3 ln (8-y) + k

Let k = ln(c)

ln | $\sqrt{1+\ u^2}$ + u | = -0.3 ln (8-y) + ln(c)

Let's make everything within the ln function:

ln | $\sqrt{1+\ u^2}$ + u | = ln($\frac{c}{(8-y)^{0.3}}$)

Taking the exp() of both sides:

Equation 2: $\sqrt{1+\ u^2}$ + u = $\frac{c}{(8-y)^{0.3}}$

Recall that u = dx/dy.

When y = 0, the missile is pointing straight up, so at that time dx/dy = u = 0. So…

$\sqrt{1+\ 0^2}$ + 0 = $\frac{c}{(8-0)^{0.3}}$

1 = c × $8^{-0.3}$

c = $8^{0.3}$

Let's put that in equation 2:

$\sqrt{1+\ u^2}$ + u = $8^{0.3} \times (8-y)^{-0.3}$

$\sqrt{1+\ u^2}$ = $8^{0.3} \times (8-y)^{-0.3}$ – u

Squaring both sides…

1 + u^2 = $8^{0.6} \times (8-y)^{-0.6}$ – 2u×$8^{0.3} \times (8-y)^{-0.3}$ + u^2

Subtract u^2 from each side:

1= $8^{0.6} \times (8-y)^{-0.6}$ – 2u×$8^{0.3} \times (8-y)^{-0.3}$

2u×$8^{0.3}$ × $(8 - y)^{-0.3}$ = $8^{0.6}$ × $(8 - y)^{-0.6}$ – 2u×$8^{0.3}$ – 1

u = (1/2) × [$8^{0.3}$ × $(8 - y)^{-0.3}$ – $8^{-0.3}$ × $(8 - y)^{0.3}$]

Recall that u = dx/dy:

dx/dy = (1/2) × [$8^{0.3}$ × $(8 - y)^{-0.3}$ – $8^{-0.3}$ × $(8 - y)^{0.3}$]

Let's move the dy to the other side:

dx = (1/2) × [$8^{0.3}$ × $(8 - y)^{-0.3}$ – $8^{-0.3}$ × $(8 - y)^{0.3}$] dy

Let's integrate both sides:

Equation 3: x = (1/2) × [$-8^{0.3} * (\frac{1}{0.7})$ × $(8 - y)^{0.7}$ + $8^{-0.3}$ × $(\frac{1}{1.3})$ × $(8 - y)^{1.3}$] + c

Where c is another constant of integration.

We're given when x=0, y=0. Let's plug that into the equation to find c:

0 = (1/2) × [$-8^{0.3} * (\frac{1}{0.7})$ × $(8 - 0)^{0.7}$ + $8^{-0.3}$ × $(\frac{1}{1.3})$ × $(8 - 0)^{1.3}$] + c

0 = (1/2) × [$-8^{0.3} * (\frac{1}{0.7})$ × $8^{0.7}$ + $8^{-0.3}$ × $(\frac{1}{1.3})$ × $8^{1.3}$] + c

0 = (1/2) × [$-8^{1} * (\frac{1}{0.7})$ + $8^{-0.3}$ × $(\frac{1}{1.3})$ × $8^{1.3}$] + c

0 = (1/2) × [-80/7 + 80/13] + c

0 = (1/2) × [(-1040/91) + (560/13)] + c

0 = (1/2) × (-480/91) + c

0 = -240/91 + c

c = 240/91

Let's put that value of c into equation (3):

x = (1/2) × [$-8^{0.3} * (\frac{1}{0.7})$ × $(8 - y)^{0.7}$ + $8^{-0.3}$ × $(\frac{1}{1.3})$ × $(8 - y)^{1.3}$] + 240/91

When the missile strikes the plane, y = 8. Let's solve for x when that happens:

x = (1/2) × [$-8^{0.3} * (\frac{1}{0.7}) \times (8-8)^{0.7} + 8^{-0.3} \times (\frac{1}{1.3}) \times (8-8)^{1.3}$] + 240/91

x = (1/2) × [$-8^{0.3} * (\frac{1}{0.7}) \times 0 + 8^{-0.3} \times (\frac{1}{1.3}) \times 0$] + 240/91

x = 240/91

So, when the missile strikes the plane, the plane will be at 240/91 miles away.

Let's solve for the time it takes the plane to travel that far:

d = r × t

(240/91) = (1/600)×t

t = 240/54600 = 6/1365 = 2/455 hours

= 120/455 = 24/91 = minutes

= 1440/91 = apx. 15.8242 seconds

Finally, solving for the distance traveled by the missile:

d = r × t

d = 2000 × (240/54600) = 4800/546 = 2400/273 = 800/91 = 8.7912 miles.

Now, for extra credit, let's prove:

$\frac{1}{\sqrt{1+u^2}}\, du = \ln |\, \sqrt{1+\, u^2} + u\, |$

Let u = tan(x), then

du = $\sec^2(x)$ dx

$$\frac{1}{\sqrt{1+u^2}}\ du =$$

$$\frac{1}{\sqrt{1+tan^2(x)}}\ sec^2(\mathrm{x})\ \mathrm{dx} =$$

$$\frac{1}{\sqrt{sec^2(x)}}\ sec^2(\mathrm{x})\ \mathrm{dx} =$$

$$\frac{1}{\sec\,(x)}\ sec^2(\mathrm{x})\ \mathrm{dx} =$$

sec(x) dx =

sec(x) * $\left(\frac{\sec(x)+\tan\,(x)}{\sec(x)+\tan\,(x)}\right)$dx =

$\left(\frac{sec^2(x)+\sec(x)*\tan\,(x)}{\sec(x)+\tan\,(x)}\right)$dx

Let v = sec(x) + tan(x)

dv = tan(x)*sec(x) + sec^2(x) dx

$\left(\frac{sec^2(x)+\sec(x)*\tan\,(x)}{\sec(x)+\tan\,(x)}\right)$dx = $\frac{1}{v}$ dv

= ln(v)

= ln(sec(x) + tan(x))

Recall, u = tan(x)

u^2 = tan^2(x)

u^2 = sec^2(x) – 1

sec^2(x) = u^2 + 1

sec(x) = $\sqrt{u^2\ +\ 1}$

So, ln(sec(x) + tan(x)) = ln $\left|\sqrt{u^2\ +\ 1} + u\right|$

Thus, we have proven

$$\frac{1}{\sqrt{1+u^2}}\ du = \ln|\sqrt{u^2\ +\ 1} + u\ |$$

I would like to thank:

- Alan Curry for his help with a solution to a similar problem.
- chevy and DogHand for correcting some typos in earlier drafts of this solution.

Image source: https://en.wikipedia.org/wiki/Surface-to-air_missile