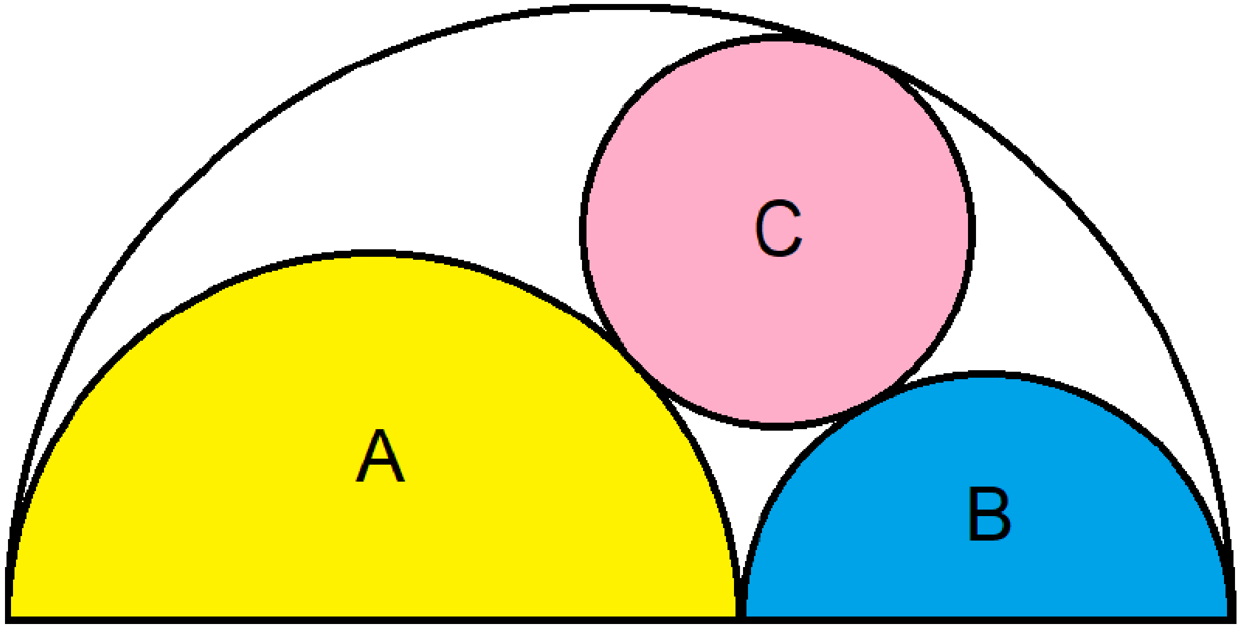
Question:



A and B are semicircles in a larger semicircle enclosing A, B, and circle C. Circle C is tangent to A, B, and the larger semicircle.

The base of A has length 6
The base of B has length 4

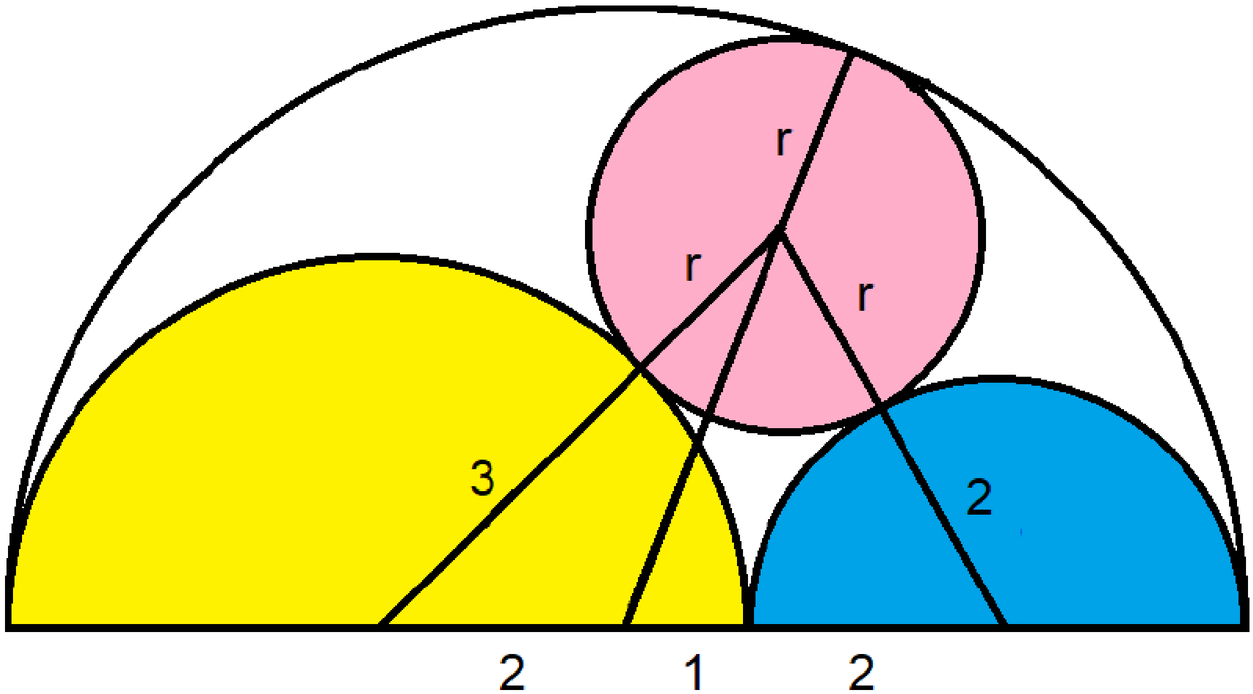
What is the radius of C?

Answer:

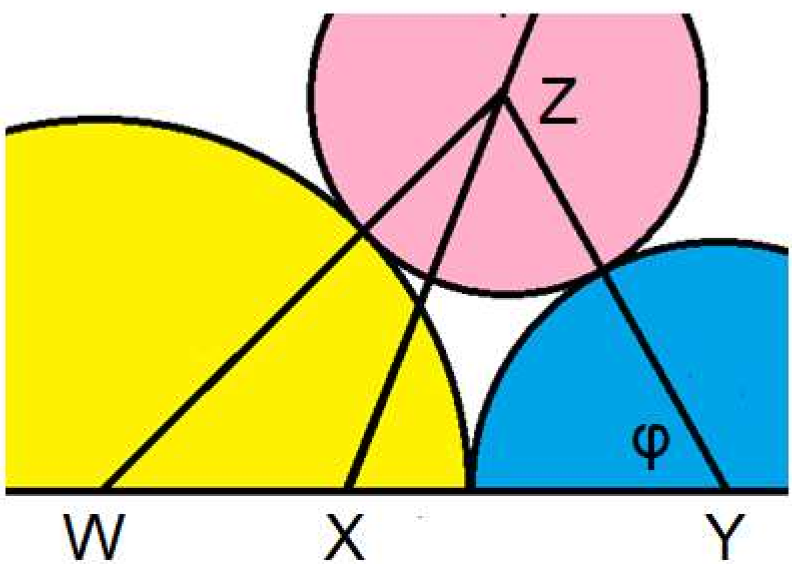
The entire base has a length of 4+6 = 10. Semicircle A has a length of 6, so the center of the base must be one unit to left of where semicircles A and B touch.

Let's define r as the radius of circle C.

Then, we can set up the following diagram.



Let's define some points in the diagram and the angle ϕ.



Triangle WYZ has sides of length r+3, 5, and r+2, where angle ϕ is opposite the side of length 3+r.

Triangle XYZ has sides of length 5-r, 3, and r+2, angle ϕ is opposite the side of length r-5.

If it's not clear why the segment XZ has length 5-r, it is because X is the center of the base of the large semicircle that encloses everything, that large semicircle has radius of 5, and the length of XZ is r less than that radius.

Next, let's review the law of cosines. That says that in any triangle with sides of length x, y, and z, and angle ϕ opposite of side x, then:

$$x^2 = y^2 + z^2 - 2yz \cdot \cos \phi$$

I will prove the law of cosines after the solution to the problem, to show no formula memorization is required to solve the problem.

Let's apply the law of cosines to triangle WYZ:

$$(r+3)^2 = 5^2 + (r+2)^2 - 2 \cdot 5 \cdot (r+2) \cos \phi$$

Factoring that out:

$$r^2 + 6r + 9 = 25 + r^2 + 4r + 4 -10 (r+2) \cos \phi$$

$$(1)\ 10 (r+2) \cos \phi = 20 - 2r$$

Let's apply the law of cosines to triangle XYZ:

$$(5-r)^2 = 3^2 + (r+2)^2 - 2 \cdot 3 \cdot (r+2) \cos \phi$$

Factoring that out:

$$r^2 - 10r + 25 = 9 + r^2 + 4r + 4 -6 (r+2) \cos \phi$$

$$(2)\ 6 (r+2) \cos \phi = -12 +14r$$

Next, let's multiply both sides of equation (1) by 3 and equation (2) by 5, to get the left sides equal.

$(3)\ 30\ (r+2) \cos \phi = 60 - 6r$

$(4)\ 30\ (r+2) \cos \phi = -60 + 70r$

Then, subtract (4) from (3):

$0 = 120 - 76r$

$76r = 120$

$r = 120/76 = 60/38 = 30/19$

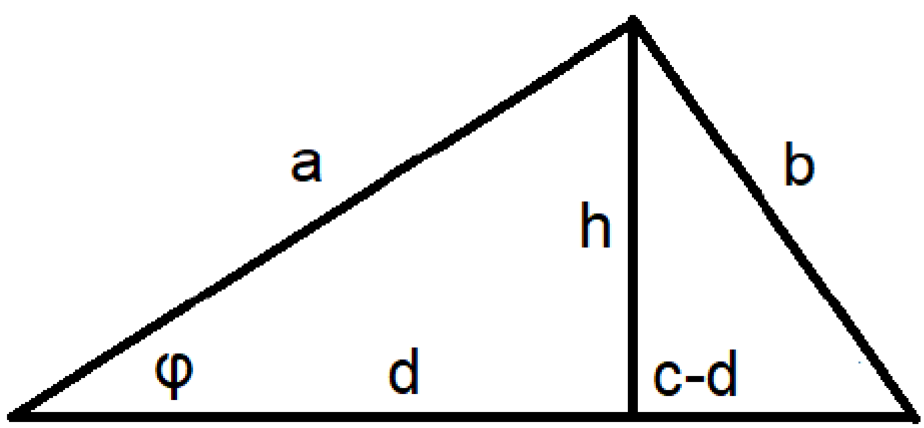
Proof of the Law of Cosines

Consider the following diagram, where:

c = length of base

d = length from left corner to point on base directly under top corner

h = height



We know:

$\cos \phi = d/a$

$\sin \phi = h / a$

Rearranging:

$d = a \cdot \cos \phi$

$h = a \cdot \sin \phi$

The Pythagorean formula tells us:

$h^2 + (c-d)^2 = b^2$

$h^2 + c^2 - 2cd + d^2 = b^2$

$a^2\sin^2\phi + c^2 - 2cd + a^2\cos^2\phi = b^2$

$a^2(\sin^2\phi + \cos^2\phi) + c^2 - 2cd = b^2$

Recall that $\sin^2\phi + \cos^2\phi = 1$

$a^2 + c^2 - 2cd = b^2$

$b^2 = a^2 + c^2 - 2cd$

$b^2 = a^2 + c^2 - 2ac \cdot \cos \phi$

Again, where a, b, and c are the sides of any triangle and ϕ is the angle opposite of side b.