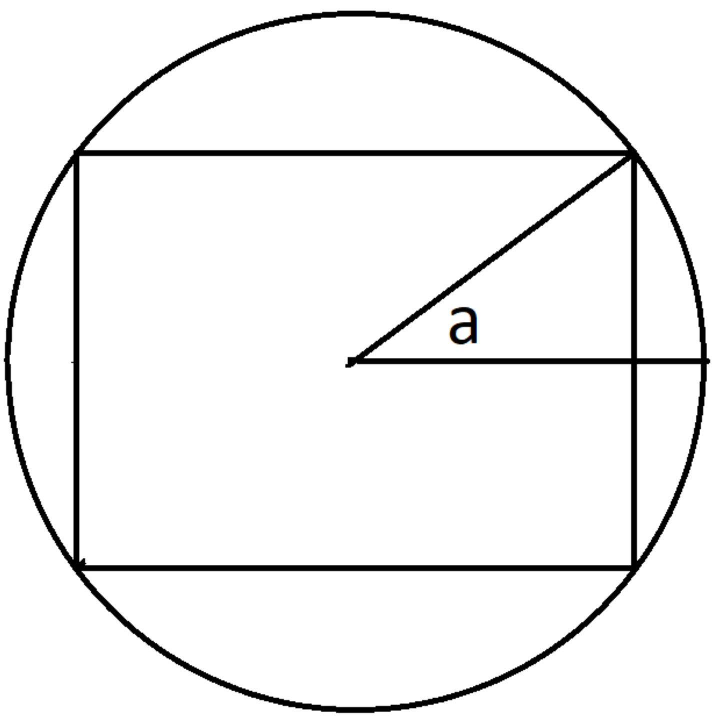
Question: What is the mean area and perimeter of a randomly drawn rectangle in a circle of radius 1?

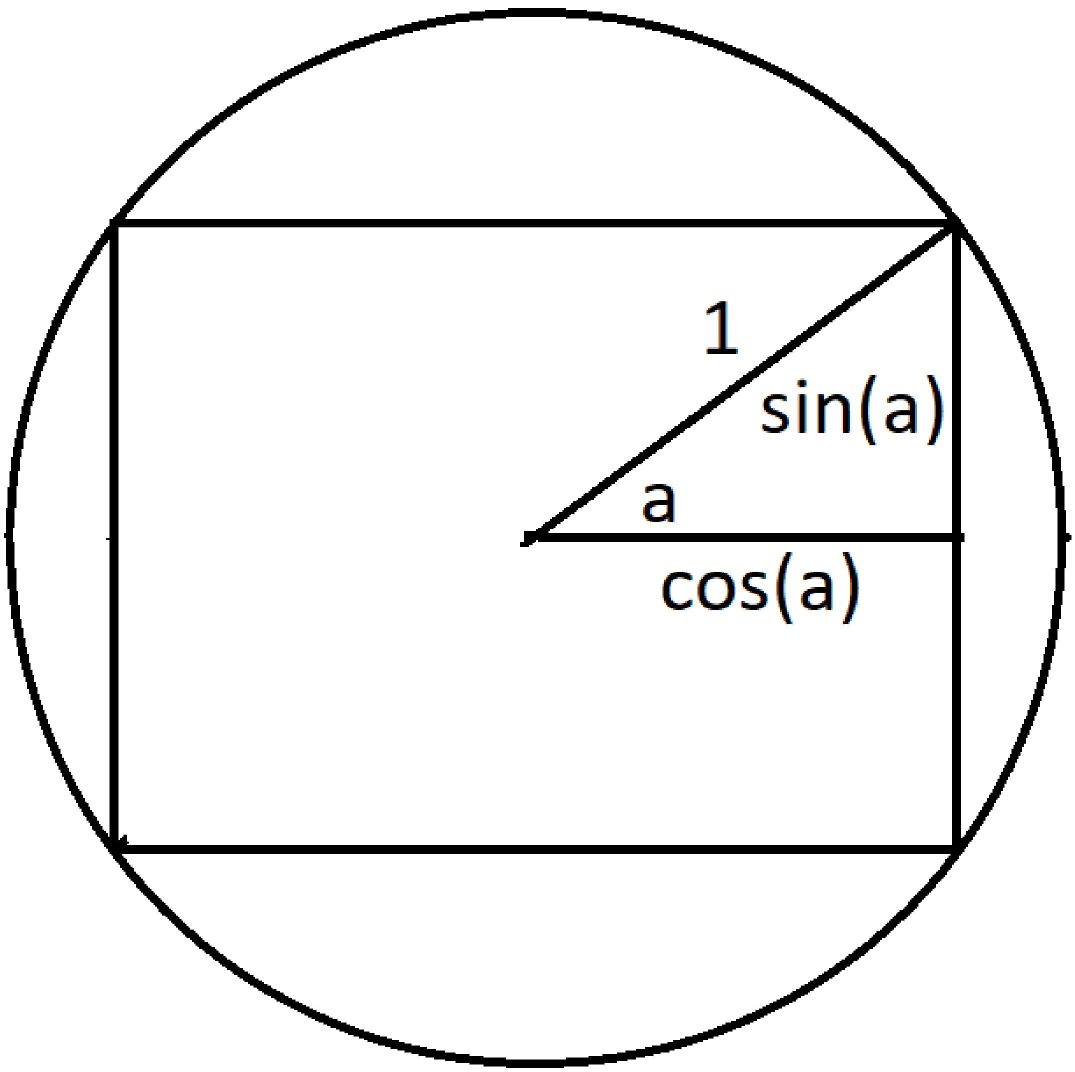
Answer: 4/pi = apx. 1.2732395447351626861510

Solution: Let's draw our random triangle by choosing a random angle (a) from 0 to 90 degrees for a point in the upper-right corner of the circle. We can then reflect that point to the bottom or the circle and left side of the circle for two more corners.

For the opposite corner we can reflect the random point across the center of the circle. Note that the two diagonals of the rectangle must go through the center of the circle.



Let's work on the area first. To begin, find the area of the triangle created by the angle a.



Simple trigonometry gives us an angle of the triangle above equal to cos(a)×sin(a)/2. There are 8 such triangles in the rectangle. Thus, given angle a, the area of the rectangle is 8×cos(a)×sin(a)/2 = 4×cos(a)×sin(a).

To get the average area of the rectangle, integrate from 0 to π/2 radians, and divide by that arc length. If you don't get that step, please take a course in integral calculus and come back. I'll wait.

We can express the average area as:

$\int_0^{\pi/2} 4\cos(a)\sin(a)\, da$ / (π/2) =

$\frac{2}{\pi}\int_0^{\pi/2} 4\cos(a)\sin(a)\, da$

Next, let's make a substitution:

b = sin(*a*)

Taking the derivative of each side:

db = cos(*a*) *da*

Plugging that into where we left off:

$\frac{2}{\pi}\int \quad 4\, b\, db \quad =$

$\frac{8}{\pi}\int \quad b\, db \quad =$

$\frac{8}{\pi} \times \frac{b^2}{2}$

Let's get back to expressing things in terms of a. Recall, b = sin (a). Resuming:

$\frac{8}{\pi} \times \frac{\sin(a)^2}{2}$ from 0 to π/2 =

$\frac{8}{\pi} \times (\frac{\sin(\pi/2)^2}{2} - \frac{\sin(0)^2}{2}) =$

$\frac{8}{\pi} \times (\frac{1^2}{2} - \frac{0^2}{2}) =$

$\frac{4}{\pi}$ = apx. 1.2732395447351626861510701069801148962756771659236515899 8…

The average perimeter of the rectangle is easier. It can be expressed as:

$\int_0^{\pi/2} 4 \times [\cos(a) + \sin(a)]\, da$ / (π/2) =

$\frac{8}{\pi} \int_0^{\pi/2} \cos(a) + \sin(a)\, da =$

$\frac{8}{\pi} \times$ [sin(a) – cos(a) for a from $\frac{\pi}{2}$ to 0] =

$\frac{8}{\pi} \times$ [(1 – 0) – (0 – 1)] = 16/π = apx. 5.09295817894065074460428042 79

Acknowledgement: Thank you to USpapergames for the problem.