Question

Consider a unit square. One line extends from the lower left corner to the upper right corner. Another line extands from the lower right corner to half way between the two upper corners. A circle is tangent to these two lines and the bottom of the square. What is the radius of the circle?

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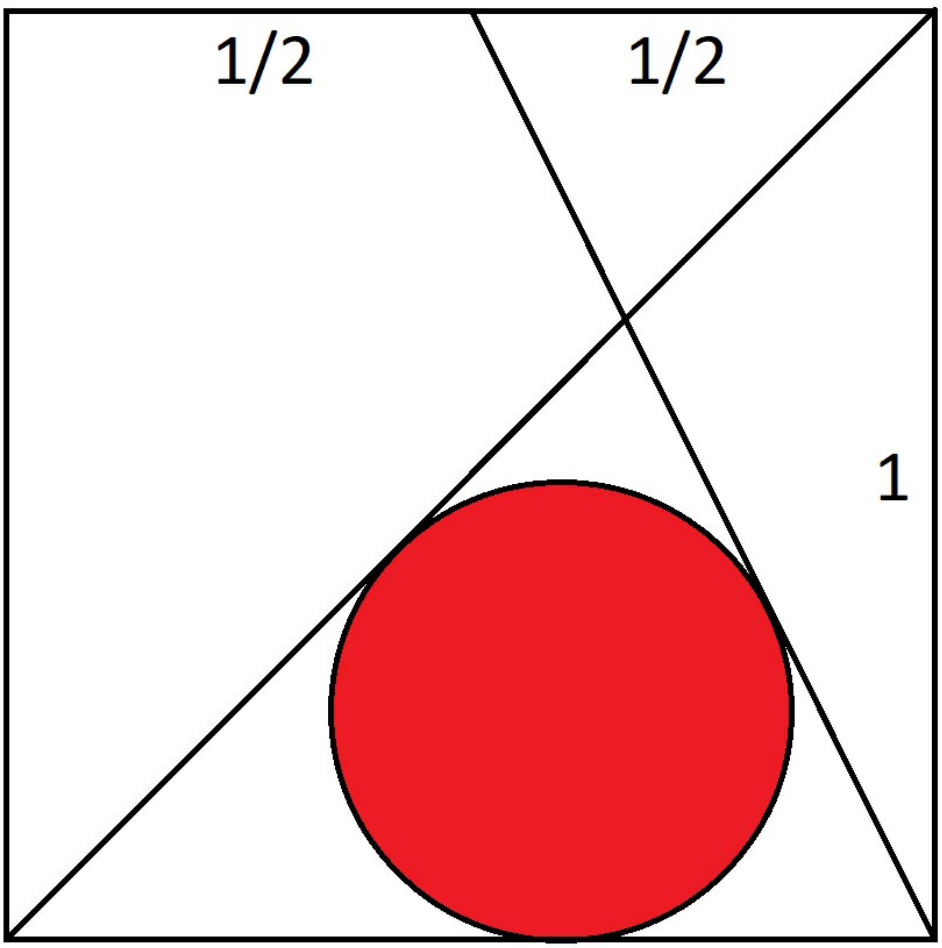


Hint:

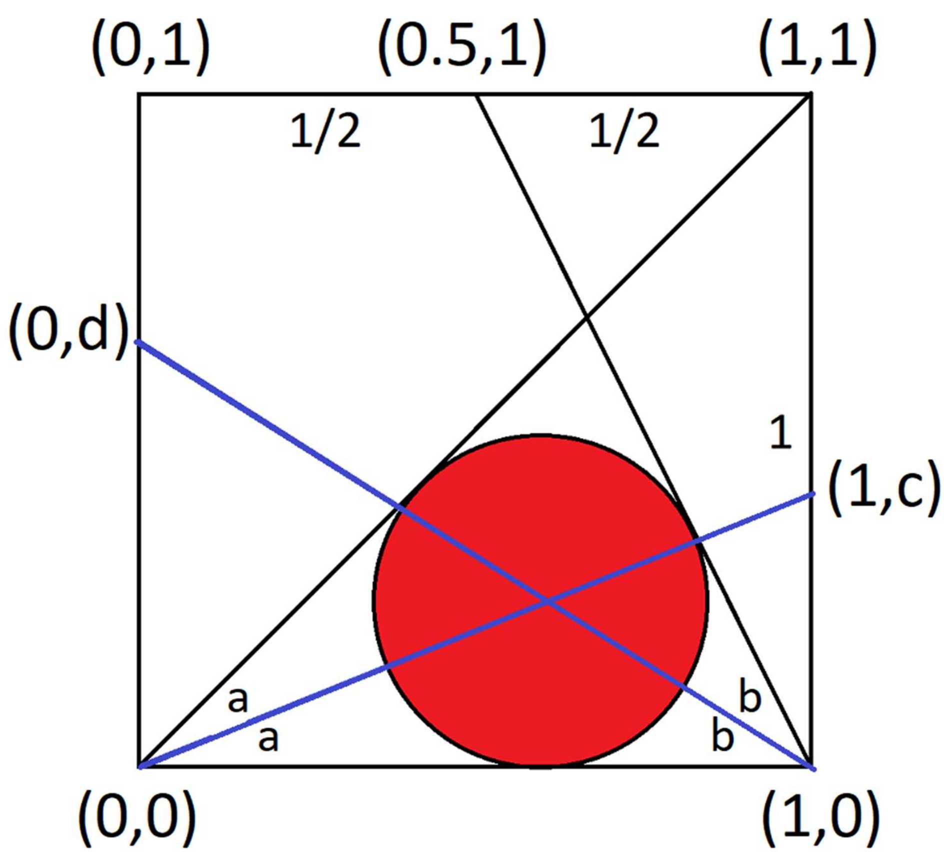
$$\tan(2x) = \frac{2\times\tan(x)}{1-\tan^2(x)}$$

Answer

First, let's assign coordinates to the corners of the square as (0,0), (0,1), (1,0), and (1,1).

Second, consider the angle between the bottom of the circle and the line that passes through (0,0) and (1,1). Then, bisect that angle. Call each of the angles formed by the bisection a.

Third, consider the angle between the bottom of the circle and the line that passes through (1,0) and (0.5,1). Then, bisect that angle. Call each of the angles formed by the bisection b.



Where the two bisection lines meet will be the center of the circle. The distance from that center to the bottom of the square will be our answer. To solve, we will find the equation of both bisection lines. To do that, we need to find angles *a* and *b*.

Tan(2a) = 1

Recall our hint:

$$\tan(2x) = \frac{2\times\tan(x)}{1-\tan^2(x)}$$

In this case:

tan(2a) = 2tan(a)/(1-tan^2(a)) = 1

Let's call the point where the line that bisects angle a and the right side of the square (1,c). Then,

tan(a) = c/1 = c

tan(2a) = 1

tan(2a) = 2tan(a)/(1-tan^2(a)) = 2c/(1-c^2) = 1

2c = 1-c^2

c^2 + 2c – 1 = 0

Using the quadratic formula:

c = (-2 +/- $\sqrt{4+4}$)/2 = (-2 + 2$\sqrt{2}$)/2 = -1 + $\sqrt{2}$ = apx. 0.4142.

That is the slope of the line that bisects angle a. The y intercept is obviously 0. This, the equation of that line is y = $(\sqrt{2}-1)x$

Let's call the point where the line that bisects angle b and the left side of the square (0,d). Then,

tan(b) = d/1 = d.

tab(2b) = 2.

$\tan(2b) = 2\tan(b)/(1-\tan^2(b)) = 2d/(1-d^2) = 2$

$2d/(1-d^2) = 2$

$2d = 2(1-d^2)$

$d = 1-d^2$

$d^2 - d - 1 = 0$

d = (-1 +/- $\sqrt{1+4}$)/2 = $(\sqrt{5}-1)$/2 = apx. 0.6180.

Two points the line that bisects angle a go through are (0,d) and (1,0). That makes the slope of that line (d-0)/(0-1) = -d = $(1-\sqrt{5})$/2.

The y intercept is d, so the equation of the line that bisects angle b is

y = $(\frac{x}{2})(1-\ \sqrt{5}) + \frac{(\sqrt{5}-1)}{2}$

Next, set the two equations for the lines equal to each other to find the x value where they cross.

$$(\frac{x}{2})(1-\ \sqrt{5}) + \frac{(\sqrt{5}-1)}{2} = (\sqrt{2}-1)x$$

$$x\ (1-\ \sqrt{5}) + (\sqrt{5}-1) = 2\ (\sqrt{2}-1)\ x$$

$$x\ (1-\ \sqrt{5} + 2 - 2\sqrt{2}) = 1-\ \sqrt{5}$$

$x = \dfrac{1-\sqrt{5}}{3-\sqrt{5}-2\sqrt{2}}$

Mathematicians abhor a radical in the denominator, so let's get rid of that. It will take two steps. First, multiply the numerator and denominator by -3 + $\sqrt{5}$ - 2$\sqrt{2}$

$$\frac{1-\sqrt{5}}{3-\sqrt{5}-2\sqrt{2}} \times \frac{-3+\sqrt{5}-2\sqrt{2}}{-3+\sqrt{5}-2\sqrt{2}} =$$

$$= \frac{-8-2\sqrt{2}+4\sqrt{5}+2\sqrt{10}}{-6+6\sqrt{5}}$$

Next, multiply both numerator and denominator by (6 + 6$\sqrt{5}$)

$$= \frac{-8-2\sqrt{2}+4\sqrt{5}+2\sqrt{10}}{-6+6\sqrt{5}} \times \frac{6\sqrt{5}+6}{6\sqrt{5}+6}$$

$$= \frac{72-1\quad\sqrt{2}-2\quad\sqrt{5}+12\sqrt{50}}{144}$$

$$= \frac{6-\sqrt{2}-2\sqrt{5}+\sqrt{50}}{12}$$

= apx. 0.5987.

Recall, one of our line equations is y = $(\sqrt{2}-1)x$

At x = $\dfrac{6-\sqrt{2}-2\sqrt{5}+\sqrt{50}}{12}$,

y = $\dfrac{1+\sqrt{2}+\sqrt{5}-\sqrt{10}}{6}$

= apx. 0.248000646617418

This y value is the distance from the center of the circle to the bottom of the square, which is one way to measure the radius of the circle. Thus, that is our answer.