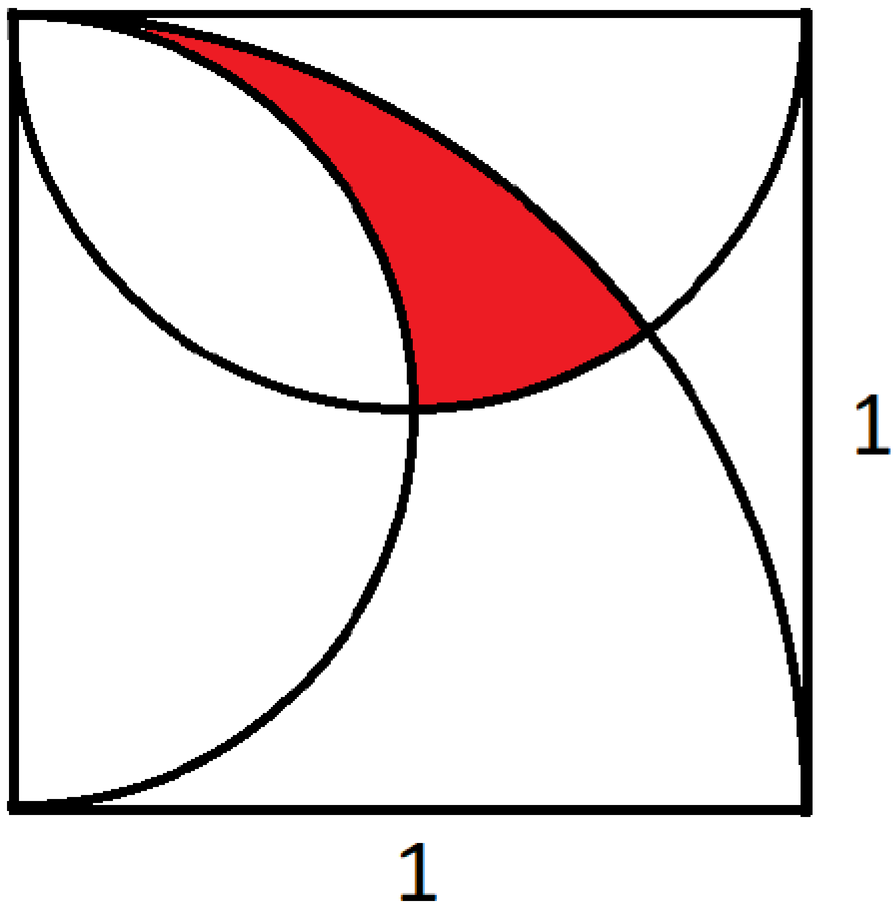
Question

In the figure below, there is a quarter-circle and two semi-circles in a square of side length one. What is the area of the red region?

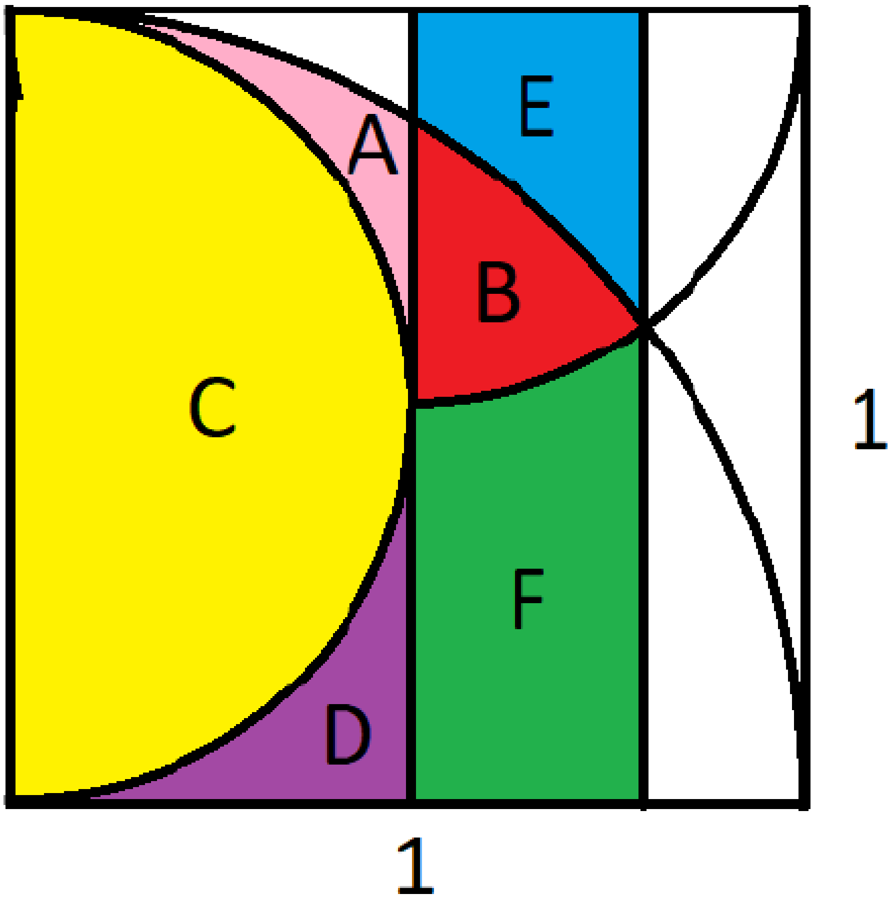


Hint

$\int \sqrt{1-x^2}\,\mathrm{dx} = \frac{1}{2}\left[\, x\sqrt{1-x^2} + \sin^{-1}(x) \,\right]$

Solution

To aid in discussion, let's label various parts of the diagram, as follows:



First, let's find the point where the upper semicircle intersects the quarter circle. In other words, there regions B, E, and F meet.

If we assign the lower left corner of the diagram coordinates (0,0), then the equation of the large circle is:

(1) $x^2 + y^2 = 1$

The equation for the small circle centered at (0.5, 1) is:

(2) $(x-0.5)^2 + (y-1)^2 = 1/4$

Let's solve for x in equation (1)

$$x^2 = 1 - y^2$$
$$x = \sqrt{1 - y^2}$$

Now for equation (2)

$$(x-0.5)^2 = \frac{1}{4} - (y-1)^2$$
$$x - \frac{1}{2} = \sqrt{\frac{1}{4} - (y-1)^2}$$
$$x = \sqrt{\frac{1}{4} - (y-1)^2} + \frac{1}{2}$$
$$x = \sqrt{\frac{1}{4} - (y^2 - 2y + 1)} + \frac{1}{2}$$
$$x = \sqrt{-y^2 + 2y - \frac{3}{4}} + \frac{1}{2}$$

Now solve for x, to find where the two curves meet:

$$\sqrt{1 - y^2} = \sqrt{-y^2 + 2y - \frac{3}{4}} + \frac{1}{2}$$

$$1 - y^2 = -y^2 + 2y - \frac{3}{4} + \sqrt{-y^2 + 2y - \frac{3}{4}} + \frac{1}{4}$$

$$1 = 2y - \frac{1}{2} + \sqrt{-y^2 + 2y - \frac{3}{4}}$$

$$\frac{3}{2} - 2y = \sqrt{-y^2 + 2y - \frac{3}{4}}$$

$$\frac{9}{4} - 6y + 4y^2 = -y^2 + 2y - \frac{3}{4}$$

$$\frac{12}{4} - 8y + 5y^2 = 0$$

$$5y^2 - 8y + 3 = 0$$

$$y = 1 \text{ or } \frac{3}{5}$$

Eyeballing the diagram, 1 clearly isn't right, so y must be 3/5.

Substitute y=3/5 in equation (1) gives us:

$$x^2 + (\frac{3}{5})^2 = 1$$

$$x^2 = 25/25 - 9/25$$

$$x^2 = 16/25$$

$$x = \frac{4}{5}$$

So, where the upper semicircle intersects the quarter circle is at point ($\frac{4}{5}, \frac{3}{5}$).

Now that we've found that, let's find the area under the quarter circle from x = 0 to 4/5. In other words A+B+C+D+F. This is where the hint comes in.

A+B+C+D+F = $\int_0^{0.8} \sqrt{1 - x^2}$ dx =

$\frac{1}{2}$ [0.8 $\sqrt{1 - 0.8^2}$ + $\sin^{-1}(0.8)$ - (0 + 0)] =

$\frac{1}{2}$ [0.48 + $\sin^{-1}(0.8)$] =~ 0.703647609000806

Next, let's find C and D, which we need to subtract out of that.

C is half of a circle of radius 0.5. Thus C = $\frac{1}{2} \bullet \pi \bullet (1/2)^2 = \frac{\pi}{8}$ =~ 0.392699081698724.

D is 0.25 less 1/4 of a circle of radius 5:

D = $\frac{1}{4}$ - $(\frac{1}{4}) \bullet \pi \bullet (\frac{1}{2})^2$ = 0.25 - $\frac{\pi}{16}$ =~ 0.0536504591506379.

Next, need to subtract F out of A+B+C+D+F.

Let's find that as (E+B+F) - (B+E) = F

To find E+B, let's invert the diagram and multiply the radius of the semicircle by 2. After doing that, we will have:

4•(E+B) = $\int_0^{0.6} \sqrt{1 - x^2}$ dx

= $\frac{1}{2}$ [0.6•$\sqrt{1 - (0.6)^2}$ + $\sin^{-1}(0.6)$ - 0 - 0]

Dividing by 4:

B+E = $\frac{1}{8}$ [0.48 + $\sin^{-1}(0.6)$] =~ 0.140437638599161

E+B+F is rectangular, so easy to find as 1 • 0.3 = 0.3.

Thus, F = 0.3 - $\frac{1}{8}$ [0.48 + $\sin^{-1}(0.6)$] = 0.159562361400839

Putting this all together:

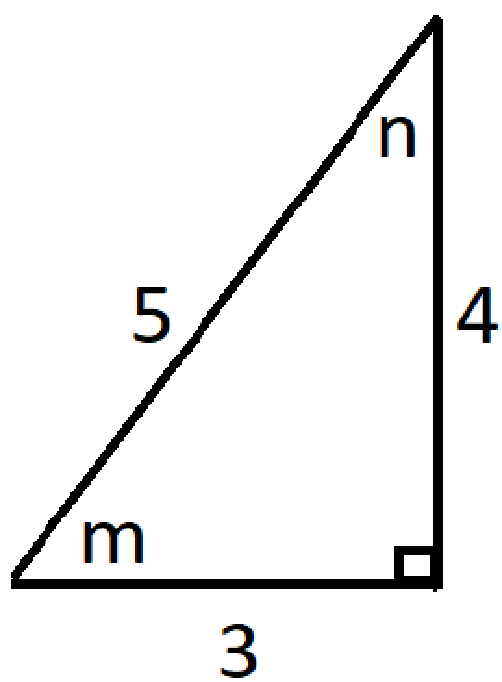
A + B = A+B+C+D+F - C - D - F

$$= \frac{1}{2} [\ 0.48 + \sin^{-1}(0.8)\ ] - \frac{\pi}{8} - (0.25 - \frac{\pi}{16}) - [0.3 - \frac{1}{8} [\ 0.48 + \sin^{-1}(0.6)\ ]]$$

$$= 0.24 + \sin^{-1}(0.8)/2 - \frac{\pi}{8} - 0.25 + \frac{\pi}{16} - 0.3 + 0.06 + \sin^{-1}(0.6)/8$$

$$= -0.25 - \frac{\pi}{16} + \sin^{-1}(0.8)/8 + \sin^{-1}(0.6)/8 + 0.375 \bullet \sin^{-1}(0.8)$$

Next, let's pause to consider the 3-4-5 triangle. Note that:



$\sin(m)=0.8$
$\sin(n)=0.6$

In other words:

$\sin^{-1}(0.8) = m$
$\sin^{-1}(0.6)=n$
$m + n = \pi/2$

Getting back to the overall answer:

$$A + B = - 0.25 - \frac{\pi}{16} + \frac{\pi}{16} + \frac{3}{8} \bullet \sin^{-1}(0.8)$$

$$= \frac{3}{8} \bullet \sin^{-1}(0.8) - \frac{1}{4}$$

$$=\sim 0.0977357067506052$$