Question:

Upon opening a Facebook account, the following are the average number of ads for fake stamps received by number of days since opening the account:

4 ads within one day.

16 ads within two days.

36 ads within three days.

What is the expected number of hours until the first ad for fake stamps appears?

Answer

$\frac{\sqrt{\pi}}{4}$ days =~ 0.443133 days

=~ 10.634723 hours

=~ 10 hours, 38.083387 minutes

=~ 10 hours, 38 minutes, 5 seconds.

Solution

It is easily seen the equation for the expected number of fake stamp ads in t days is $4t^2$.

Recall, the Poisson distribution says that if the expected number of events to occur in a period of time is μ, then the probability of exactly x events is $\frac{e^{-\mu}\mu^x}{x!}$.

So, the probability of zero events in a period of time μ is $\frac{e^{-\mu}\mu^0}{0!} = e^{-\mu}$.

Next, recall the expected time for an event to occur is the integral of time t from 0 to infinity of the probability that the event has not occurred yet by time t.

In our case, the probability of zero fake stamp ads by time t is e^{-4t^2}, because $4t^2$ is the expected number of fake stamp ads after t days.

In this case, the expected waiting time can be expressed as:

$$\int_0^\infty e^{-4t^2}\,\mathrm{dt}$$

At this point, you can cheat and use an integral calculator. I prefer the one at www.integral-calculator.com/. This will give you an answer of $\frac{\sqrt{\pi}}{4}$.

This equals 0.443133 days

=~ 10.634723 hours

=~ 10 hours, 38.083387 minutes

=~ 10 hours, 38 minutes, 5 seconds.

However, some of you may know by heart that:

$$\int_{-\infty}^{\infty} e^{-x^2}\ \mathrm{dx} = \sqrt{\pi}.$$

This, I think, is one of the most beautiful equations in mathematics, up there with Euler's equation of $e^{-\pi i} + 1 = 0$.

If we can assume that as correct, then substitute

y = 2t

so:

t = y/2

dt = ½ dy

$$\int_{0}^{\infty} e^{-4t^2}\ \mathrm{dt} = \frac{1}{2}\int_{0}^{\infty} e^{-y^2}\ \mathrm{dy} = \frac{\sqrt{\pi}}{4}.$$