Question: What is the expected number of rolls of a fair six-side die for any one side to be rolled six times?

Answer: 2597868106693535971 / 131621703842267136 =~19.73738396371749.

Solution:

Imagine instead of rolling a die, there are six light bulb sockets. They are always on and always have a light bulb inside. If a light bulb burns out, it is immediately replaced with a new one. There is an infinity supply of light bulbs. All light bulbs have a life expectancy of six days distributed exponentially, meaning they have a memoryless property where the probability of failing is always the same, regardless of how long the bulb has already been on. What is the expected number of days until any one socket has gone through six light bulbs?

The answer will be the same. Briefly, this is because the expected life of any given light bulb is the same as the expected rolls for any given side to appear.

The Poisson distribution says that the probability of exactly n light bulbs burning out in any one socket in x days is exp(-x/6)*(x/6)^x/x!

The following shows the probability of exactly 0 to 5 bulbs burning out in x days:

0 bulbs: exp(-x/6)*(x/6)0/0! = exp(-x/6)

1 bulb: exp(-x/6)*(x/6)1/1! = exp(-x/6)*(x/6)

2 bulbs: exp(-x/6)*(x/6)2/2! = exp(-x/6)*x2/72

3 bulbs: exp(-x/6)*(x/6)3/3! = exp(-x/6)*x3/1296

4 bulbs: exp(-x/6)*(x/6)4/4! = exp(-x/6)*x4/31104

5 bulbs: exp(-x/6)*(x/6)5/5! = exp(-x/6)*x5/933120

Adding these up, the probability that five or less bulbs have burned out in any one socket in x days is:

exp(-x/6)*(1+x/6+x^2/72+x^3/1296+x^4/31104+x^5/933120)

Taking this to the fifth power gives us the probability that five or less bulbs have burned out in five different sockets in x days is:

[exp(-x/6)*(1+x/6+x^2/72+x^3/1296+x^4/31104+x^5/933120)]^5

The probability that in x days, five specific sockets have had five or less bulbs burn out and one socket is at exactly five bulbs burned out is:

f(x) = [exp(-x/6)*(1+x/6+x^2/72+x^3/1296+x^4/31104+x^5/933120)]^5 * exp(-x/6)*x^5/933120

The expected number of days for the first socket to have six light bulbs burn out is:

$$\int_0^{\infty} 6x \times \frac{1}{6} \times f(x)dx$$

The 6 is because any one of the sockets could be the one that sees the sixth bulb blow out first.

The x represents the number of days.

The 1/6 is because that is the probability any one bulb blows out at any given time.

So, here is the whole expression in text:

x*(exp(-x/6)*(1+x/6+x^2/72+x^3/1296+x^4/31104+x^5/933120))^5*exp(-x/6)*(x^5/933120)

Here is how it should look in math book form, after simplifying it:

$$\frac{x^6\left(\frac{x^5}{933120}+\frac{x^4}{31104}+\frac{x^3}{1296}+\frac{x^2}{72}+\frac{x}{6}+1\right)^5 \mathrm{e}^{-x}}{933120}$$

Put the integral into an integral calculator, like the one at www.integral-calculator.com. Remember to put in the bounds of integration from 0 to ∞.

That will give the answer of 2597868106693535971 / 131621703842267136 =~19.73738396371749