Question: The “All” bet in craps wins if every possible total, except seven, are rolled before a total of seven is rolled. My question is when this bet does win, what is the average number of rolls needed to win?

Answer: 20.989727251191…

Solution:

Let’s use calculus to solve this. Assume the time between rolls is distributed according to the exponential distribution with a mean of one. Given a roll has happened, let’s assume the probability of the specific total are the same as with dice. The final answer will work out the same as if rolls always happened with the same time between each roll.

The following table shows the average time it takes for any given total to be rolled.

TOTAL	AVERAGE TIME
2	36
3	18
4	12
5	9
6	7.2
7	6
8	7.2
9	9
10	12
11	18
12	36

Before going further, let’s use the notation pr(x) to mean the probability of rolling to total of x.

Let’s first answer the probability that the player wins this bet with the final roll of a 2. For this to happen, the player must have already rolled totals of everything except 2 and 7 before rolling a 2. The probability, at any given time t, of this happening can be expressed as:

f(t) = pr(3) * pr(4) * pr(5) * pr(6) * pr(8) * pr(9) * pr(10) * pr(11) * pr(12) * pr(no 7) * 1/36 =

f(t) = pr(3) * $(pr(4))^2$ * $(pr(5))^2$ * $(pr(6))^2$ * pr(no 7) * 1/36 =

$$f(t) = \left(1 - e^{\frac{-t}{36}}\right) \times \left(1 - e^{\frac{-t}{18}}\right)^2 \times \left(1 - e^{\frac{-t}{12}}\right)^2 \times \left(1 - e^{\frac{-t}{9}}\right)^2 \times \left(1 - e^{\frac{-t}{7.2}}\right)^2 \times \left(e^{\frac{-t}{6}}\right)^2 \times \frac{1}{36}$$

To find the probability of this ever happening, integrate over t from one to infinity:

$$\int_0^{\infty} \left(1 - e^{\frac{-t}{36}}\right) \times \left(1 - e^{\frac{-t}{18}}\right)^2 \times \left(1 - e^{\frac{-t}{12}}\right)^2 \times \left(1 - e^{\frac{-t}{9}}\right)^2 \times \left(1 - e^{\frac{-t}{7.2}}\right)^2 \times \left(e^{\frac{-t}{6}}\right)^2 \times \frac{1}{36}\,dt$$

Let’s write that probability in text form and run it through an integral calculator. I recommend the one at www.integral-calculator.com. Here is that probability in text:

f(t) = (1-exp(-t/36))*(1-exp(-2t/36))^2*(1-exp(-3t/36))^2(1-exp(-4t/36))^2*(1-exp(-5t/36))^2*exp(-t/6)*exp(-t/36)*(1/36)

Remembering to put in the limits of integration of 0 to infinity, we get an answer of 137124850157/144403552893600 = 0.000949594711551432.

That gives us only the probability of winning. Next, let’s find how many rolls go into such a winning event. To get at that, let’s throw a t into the f(t) function above. If we integrate over that function, it will give us the average time when the bet wins and a 0 when it doesn’t. So, we integrate the following:

$$\int_0^\infty (1-e^{\frac{-t}{36}}) \times \left(1-e^{\frac{-t}{18}}\right)^2 \times \left(1-e^{\frac{-t}{12}}\right)^2 \times \left(1-e^{\frac{-t}{9}}\right)^2 \times \left(1-e^{\frac{-t}{7.2}}\right)^2 \times \left(e^{\frac{-t}{6}}\right)^2 \times \frac{1}{36} \times \text{t dt}$$

Here is the function in text form:

g(t)=(1-exp(-t/36))*(1-exp(-2t/36))^2*(1-exp(-3t/36))^2(1-exp(-4t/36))^2*(1-exp(-5t/36))^2*exp(-t/6)*exp(-t/36)*(t/36).

Remembering to put in the limits of integration of 0 to infinity, we get an answer of 150695431/226337857200 = 0.000665798611262986.

To get the average rolls required when the bet does win, we divide the overall rolls required by the probability of winning, as follows:

0.000665798611262986 / 0.000949594711551432 = 22.5002779738653

There is the first step, the average rolled required when the player wins on a total of 2.

Using the same logic, here are expressions of the rolls required for totals of 3, 4, 5, and 6.

Total of 3:

$$\int_0^\infty (1-e^{\frac{-t}{36}})^2 \times \left(1-e^{\frac{-t}{18}}\right)^1 \times \left(1-e^{\frac{-t}{12}}\right)^2 \times \left(1-e^{\frac{-t}{9}}\right)^2 \times \left(1-e^{\frac{-t}{7.2}}\right)^2 \times \left(e^{\frac{-t}{6}}\right)^2 \times \frac{1}{18} \text{dt} /$$
$$\int_0^\infty (1-e^{\frac{-t}{36}})^2 \times \left(1-e^{\frac{-t}{18}}\right)^1 \times \left(1-e^{\frac{-t}{12}}\right)^2 \times \left(1-e^{\frac{-t}{9}}\right)^2 \times \left(1-e^{\frac{-t}{7.2}}\right)^2 \times \left(e^{\frac{-t}{6}}\right)^2 \times \frac{t}{18}$$

= 0.0141877772566115 / 0.000665798611262986 = 21.3094125109363

Total of 4:

$$\int_0^\infty (1-e^{\frac{-t}{36}})^2 \times \left(1-e^{\frac{-t}{18}}\right)^2 \times \left(1-e^{\frac{-t}{12}}\right)^1 \times \left(1-e^{\frac{-t}{9}}\right)^2 \times \left(1-e^{\frac{-t}{7.2}}\right)^2 \times \left(e^{\frac{-t}{6}}\right)^2 \times \frac{1}{12} \text{dt} /$$
$$\int_0^\infty (1-e^{\frac{-t}{36}})^2 \times \left(1-e^{\frac{-t}{18}}\right)^2 \times \left(1-e^{\frac{-t}{12}}\right)^1 \times \left(1-e^{\frac{-t}{9}}\right)^2 \times \left(1-e^{\frac{-t}{7.2}}\right)^2 \times \left(e^{\frac{-t}{6}}\right)^2 \times \frac{t}{12}$$

= 0.0093528600005717 / 0.000463902392085596 = 20.1612670254262

Total of 5:

$$\int_0^{\infty}(1-e^{\frac{-t}{36}})^2 \times \left(1-e^{\frac{-t}{18}}\right)^2 \times \left(1-e^{\frac{-t}{12}}\right)^2 \times \left(1-e^{\frac{-t}{9}}\right)^1 \times \left(1-e^{\frac{-t}{7.2}}\right)^2 \times \left(e^{\frac{-t}{6}}\right)^2 \times \frac{1}{9}\,\text{dt} \,/$$
$$\int_0^{\infty}(1-e^{\frac{-t}{36}})^2 \times \left(1-e^{\frac{-t}{18}}\right)^2 \times \left(1-e^{\frac{-t}{12}}\right)^2 \times \left(1-e^{\frac{-t}{9}}\right)^1 \times \left(1-e^{\frac{-t}{7.2}}\right)^2 \times \left(e^{\frac{-t}{6}}\right)^2 \times \frac{t}{9}$$

= 0.0061727978345964 / 0.000323290973445772 = 19.0936287790659

Total of 6:

$$\int_0^{\infty}(1-e^{\frac{-t}{36}})^2 \times \left(1-e^{\frac{-t}{18}}\right)^2 \times \left(1-e^{\frac{-t}{12}}\right)^2 \times \left(1-e^{\frac{-t}{9}}\right)^2 \times \left(1-e^{\frac{-t}{7.2}}\right)^1 \times \left(e^{\frac{-t}{6}}\right)^2 \times \frac{1}{7.2}\,\text{dt} \,/$$
$$\int_0^{\infty}(1-e^{\frac{-t}{36}})^2 \times \left(1-e^{\frac{-t}{18}}\right)^2 \times \left(1-e^{\frac{-t}{12}}\right)^2 \times \left(1-e^{\frac{-t}{9}}\right)^2 \times \left(1-e^{\frac{-t}{7.2}}\right)^1 \times \left(e^{\frac{-t}{6}}\right)^2 \times \frac{t}{7.2}$$

= 0.0040993074091174 / 0.000226265359752434 = 18.1172558344883

For totals of 8 to 12, use the same answers for totals of 2 to 6, as the probability of rolling a total of 7+x is the same as 7-x.

Now, we have all the pieces. To get the overall answer, take the weighted average of the rolls required according to probability of winning with each final total.

The following table shows the average rolls required and probability of winning for all final winning totals. Note that the sum of the probability column gives us the overall probability of winning the All bet of 0.00525770. The weighted probability column shows the probability of winning that the given final total, given that there was a win.

LAST TOTAL	AVERAGE ROLLS	PROBABILITY	WEIGHTED PROBABILITY	EXPECTED ROLLS
2	22.50027797	0.00094959	0.18061015	4.06377852
3	21.30941251	0.00066580	0.12663296	2.69847390
4	20.16126703	0.00046390	0.08823288	1.77888672
5	19.09362878	0.00032329	0.06148900	1.17404816
6	18.11725583	0.00022627	0.04303501	0.77967633
8	18.11725583	0.00022627	0.04303501	0.77967633
9	19.09362878	0.00032329	0.06148900	1.17404816
10	20.16126703	0.00046390	0.08823288	1.77888672
11	21.30941251	0.00066580	0.12663296	2.69847390
12	22.50027797	0.00094959	0.18061015	4.06377852
TOTAL		0.00525770	1.00000000	20.98972725

The expected rolls column is the product of the average rolls and weighted probability. The sum of that total is our answer, 20.989727251191 rolls.