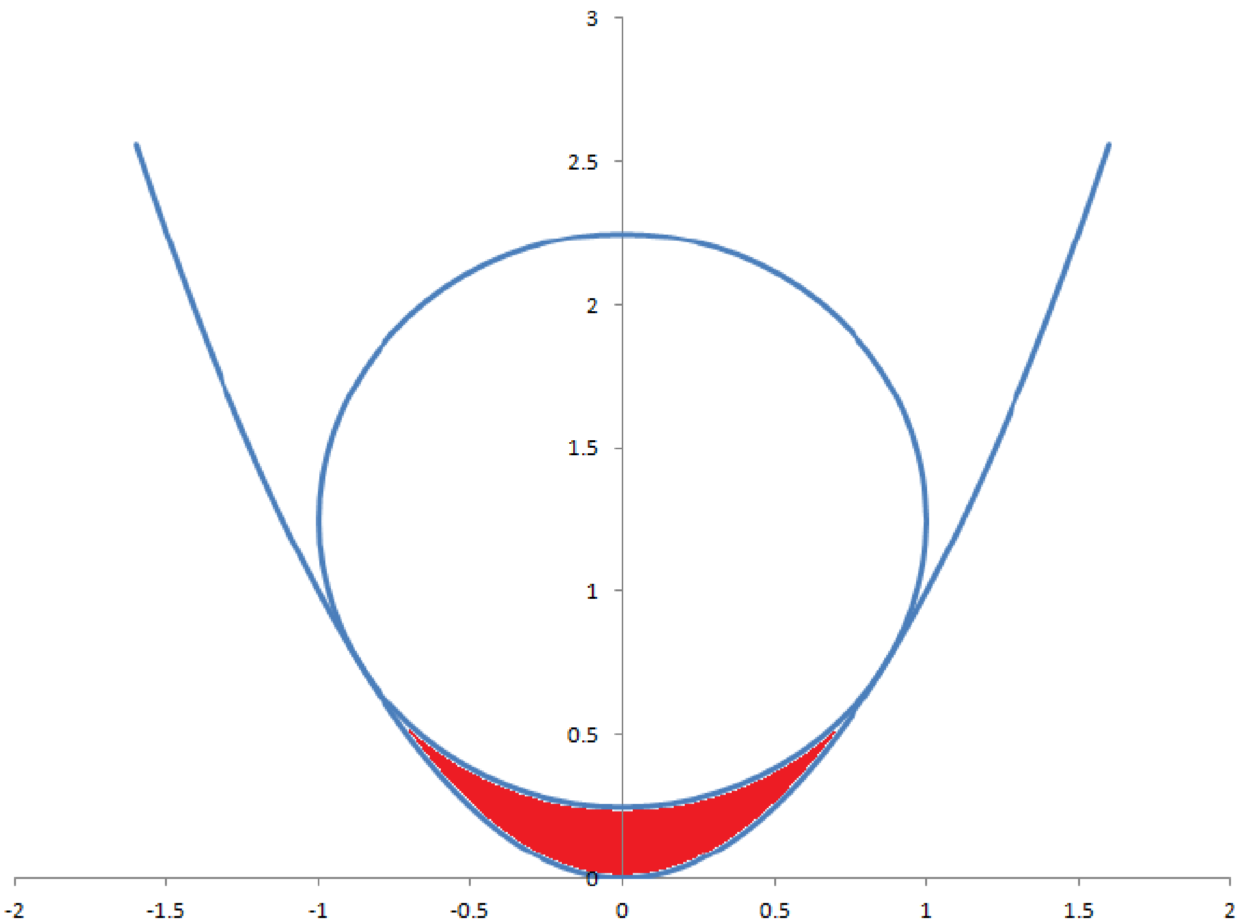
A circle of radius 1 is tangent to a parabola with equation $y=x^2$. What is the area in the red region, between the circle and parabola?



Hint:

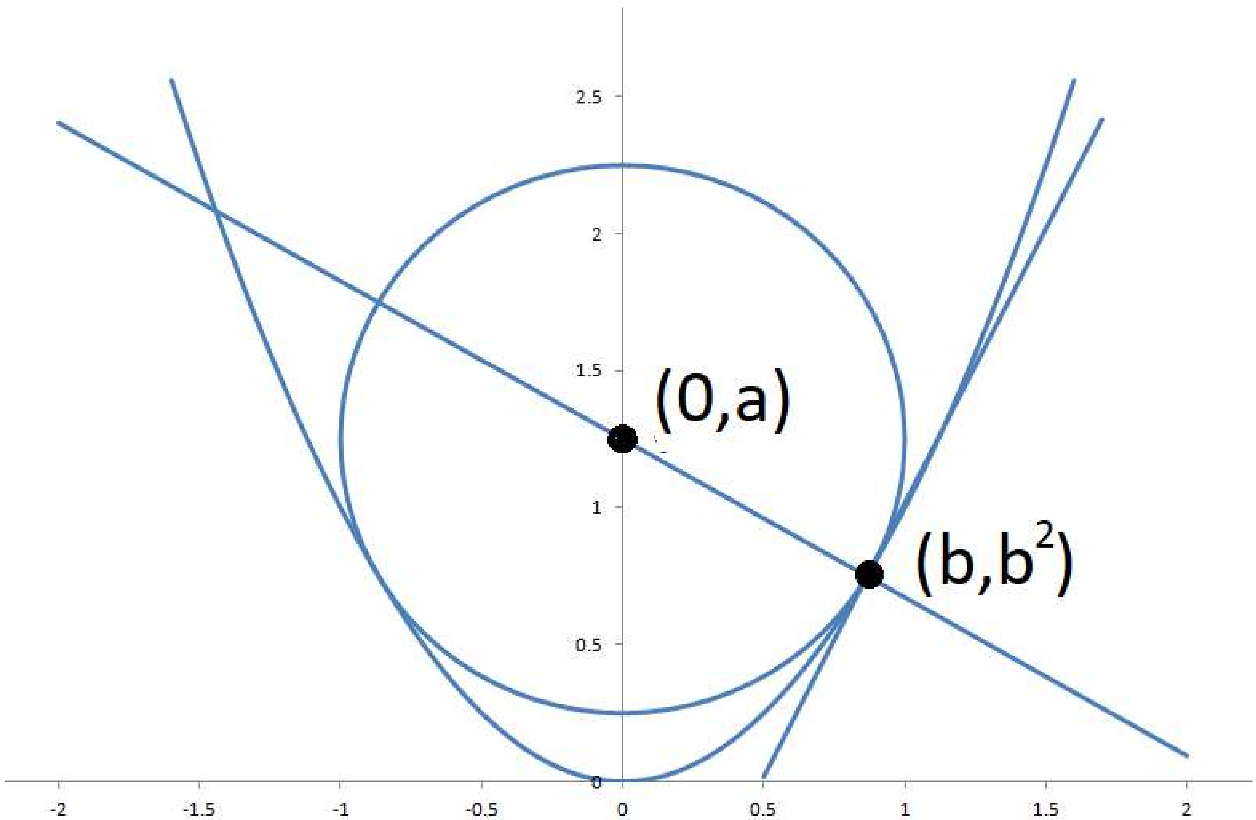
Here is an integral I use, which comes up a lot in problems involving circles:

$$\int \sqrt{1-x^2}\ dx = \frac{1}{2}[x\sqrt{1-x^2} + \sin^{-1}(x)]$$

Here are a couple others, which you could use instead, combined with a variable substitution:

$$\int \sin^2 x\ dx = \frac{x}{2} - \frac{\sin(2x)}{4} \quad \text{or} \quad \int \cos^2 x\ dx = \frac{x}{2} + \frac{\sin(2x)}{4}$$

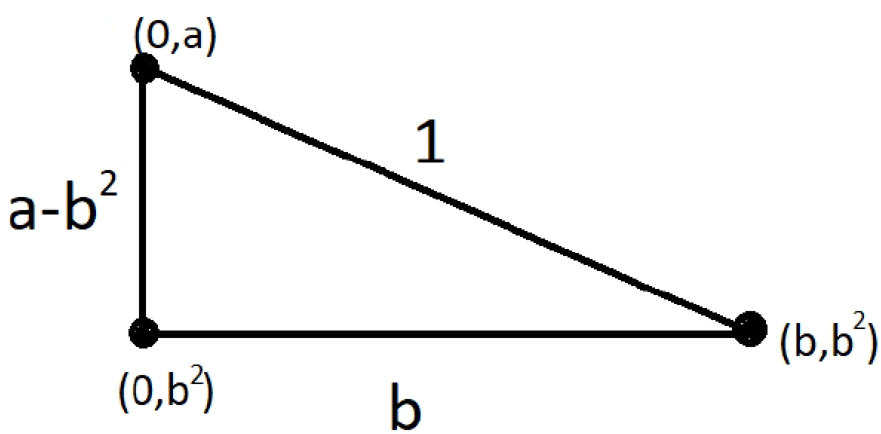
First, let's define some points on the graph, as follows:



(b,b^2) is one of the two places where the circle is tangent to the parabola.

(a,0) is the center of the circle.

Let's focus on the right triangle between the points (0,a), (b, b^2), and $(b^2,0)$:



Next, let's solve for a and b.

The Pythagorean formula tells us:

(1) $(a-b^2)^2 + b^2 = 1$

Remember the equation of the parabola is $y=x^2$. Taking the derivative will give us the slope of the line tangent to the parabola. This derivative is $y=2x$. So, the slope of the tangent line at $(b,b^2) = 2b$.

Perpendicular to this line is a line that goes through the center of the circle, like the spoke of a wheel. This spoke must have slope $-\frac{1}{2b}$. Let's find the equation of that line.

$y = -\frac{x}{2b} + a$

Let's plug in the center of the circle into this equation to solve for b:

$b^2 = -\frac{b}{2b} + a$

(2) $a = b^2 - \frac{1}{2}$

Let's substitute the value of a in equation (2) into equation (1):

$((b^2 - \frac{1}{2}) - b^2)^2 + b^2 = 1$

$1/4 + b^2 = 1$

$b^2 = 3/4$

$b = \frac{\sqrt{3}}{2}$

Equation (2) then tells us a = 3/4 - 1/2 = 1/4. So, the bottom of the circle is at (1/4,0) and the top at (5/4,0).

Next, let's find the area under the parabola.

$\int_{-\sqrt{3}/2}^{\sqrt{3}/2} x^2 \; dx$ =

$\frac{x^3}{3}$ from $\sqrt{3}/2 \; to - \sqrt{3}/2$ =

$\frac{\sqrt{3}}{4}$ =~ 0.4330

The area under the circle will be trickier. To keep the integral as simple as possible, let's first calculate the area under a unit circle, centered at (0,0) from -b to b.

Here you should know the integral:

$\int \sqrt{1-x^2} \; dx$ = $\frac{1}{2}$ [x $\sqrt{1-x^2}$ + sin^{-1} (x)]

You can also do a trig substitute where x = sin θ, but that just leads to another integral you'll have to look up or memorize, so I prefer to not add the extra step.

$\int_{-\sqrt{3}/2}^{\sqrt{3}/2} \sqrt{1-x^2} \; dx$ =

$\frac{1}{2}$ [x $\sqrt{1-x^2}$ + sin^{-1} (x)] from $\sqrt{3}/2 \; to - \sqrt{3}/2$ =

$\frac{\sqrt{3}}{4} + sin^{-1}\,(\frac{\sqrt{3}}{2})$

$\frac{\sqrt{3}}{2}$ should look familiar. Everybody should memorize the 30-60-90 (or π/6, π/3, π/2) triangle. The sides are 1, $\sqrt{3}$, and 2. Thus:

sin(π/6) = cos(π/3) = 1/2

sin(π/3) = cos(π/6) = $\frac{\sqrt{3}}{2}$

That said, getting back to the area under the circle, we have $\frac{\sqrt{3}}{4} + \frac{\pi}{3}$ =~ 1.4802

To find the area above the circle and the line y=1, consider the rectangular area and subtract the area of the circle from it:

$$2 \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{3} = \frac{3\sqrt{3}}{4} - \frac{\pi}{3} =\sim 0.2518.$$

Next, find the rectangular area directly under the circle. Since the center of the circle is at (0, 5/4), the bottom must bet at (0, 1/4).

$$2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{4} =\sim 0.4330.$$

So, the entire area under the circle is:

$$\frac{3\sqrt{3}}{4} - \frac{\pi}{3} + \frac{\sqrt{3}}{4} = \sqrt{3} + \frac{\pi}{3} =\sim 0.6849$$

So, the area between the circle and parabola equals:

$$\sqrt{3} - \frac{\pi}{3} - \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{4} - \frac{\pi}{3} =\sim 0.2518.$$

Note that this is the same as the area above the circle over the same bounds. Interesting.