Question

A bus arrives to a certain bus stop every ten minutes, exactly. At any given time, the probability of a passenger arriving is equal and independent of the time since the last passenger arrived. On average, one passenger arrives per minute. When a bus arrives, all waiting passengers get on.

You arrive at the bus stop and there are 12 passengers waiting. You have no way of telling the time. What is the expected time until the next bus arrives?

Answer

The answer is 1.54705085… minutes =~ One minute and 33 seconds.

Solution

Let's change the time between bus arrivals to every one unit of time. When we're done, we'll multiply by ten.

Let's call t the time since the last arrival. So, for any t, the expected number of waiting passengers is 10t.

Per the Poisson distribution, the probability of exactly 12 passengers at time t is:

$$\frac{e^{-10t}(10t)^{12}}{12!}$$

Of course, it's more likely the 12th passenger comes along close to the arrival of the next bus. However, it could happen anytime. An expression for the expected time when the 12th passenger arrives is the product of the integral over time from 0 to 1 of the probability of exactly 12 passengers and t. In other words:

$$\int_0^1 t\ \frac{e^{-10t}(10t)^{12}}{12!}\ dt$$

We could solve this using integral by parts, but it would take 12 steps. This would be very tedious and subject to error. So forgive me if I use an integral calculator to get 0.017619625059.

Next, we need to tease out the t factor, to get an expression for the probability of the 12th passenger arriving at a random time between 0 and 1. This can be expressed as:

$$\int_0^1 \frac{e^{-10t}(10t)^{12}}{12!}\,dt$$

Using an integral calculator, we get 0.020844352361.

Assuming a 12th passenger arrives at sometime between 0 and 1, the expected time is the ratio of the first integral to the second:

0.017619625059/0.020844352361 =~ 0.845294915129.

Now, let's multiply that by 10, since busses arrive every ten minutes. This tells us that on average, the 12th passenger arrives after 8.45294915129 minutes. Subtracting that from 10 tells us the average waiting time until the bus is 10 - 0.845294915129 = 1.547050848714 minutes.

This equals about one minute and 32.8 seconds.

Links

Integral Calculator: www.integral-calculator.com