Question:

What is the expected value of this game?

1. A deck begins with one white and one black card.
2. You draw a card at random.
3. If you draw a black card, the game ends and you're paid $1 for each black card in the deck, including the one you just drew.
4. If you drew a white card, it is added back to the deck along with another black card. Then go back to rule 2.

Hint: $\sum_{n=0}^{\infty} 1/n!$ = e

Answer:

Expected value = 1×(1/2) + 2× (1/2) × (2/3) + 3× (1/2) × (1/3) × (3/4) + 4× (1/2) × (1/3) × (1/4) × (4/5) + ...
= $1^2/2! + 2^2/3! + 3^2/4! + 4^2/5! + \ldots$
= $\sum_{n=1}^{\infty} \frac{n^2}{(n+1)!}$
Here comes the humdinger, we want to express n^2 in a way that will divide with (n+1)! or down to a 1.
= $\sum_{n=1}^{\infty} \frac{n(n+1)-n}{(n+1)!}$
(1) = $\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+1)!}$ - $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$
(2) $\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+1)!} = \sum_{n=1}^{\infty} \frac{n}{n!} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \sum_{n=0}^{\infty} \frac{1}{n!}$ = e (from the hint)
Getting back to the humdinger, we want to transform that n into a term with (n+1).
(3) $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \sum_{n=1}^{\infty} \frac{n+1}{(n+1)!}$ - $\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$
(4) $\sum_{n=1}^{\infty} \frac{n+1}{(n+1)!} = \sum_{n=1}^{\infty} \frac{1}{n!}$ = e-1 (It's e-1, not e, because the series starts with 1. i.e. $\frac{1}{0!}$ = 1)
(5) $\sum_{n=1}^{\infty} \frac{1}{(n+1)!} = \sum_{n=2}^{\infty} \frac{1}{n!}$ = e-2 (Here we miss the first two terms of $\sum_{n=0}^{\infty} 1/n!$, which both equal 1, so we subtract 2 from e.
Substituting the result from equations (4) and (5) into (3) gives us:
(6) $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$ = (e-1) - (e-2) = 1
Substituting the result from equations (2) and (6) into (1) gives us:
$\sum_{n=1}^{\infty} \frac{n^2}{(n+1)!}$ = e - 1