Question

Let:

$a^2 - bc = 5$

$b^2 - ac = 7$

$c^2 - ab = 9$

Solve for a, b and c.

Answer

Solution 1:

a = $-\sqrt{\frac{1805}{315}}$ =~ -2.393775

b = $\frac{2}{19}\sqrt{\frac{1805}{315}}$ =~ 0.251976

c = $\frac{23}{19}\sqrt{\frac{1805}{315}}$ =~ 2.897728

Solution 2:

a = $-\sqrt{\frac{1805}{315}}$ =~ 2.393775

b = $\frac{2}{19}\sqrt{\frac{1805}{315}}$ =~ -0.251976

c = $\frac{23}{19}\sqrt{\frac{1805}{315}}$ =~ -2.897728

Solution

The humdinger to this solution is to get the terms on the left side to telescope and cancel each other out. This can be done two ways. First, let's label the equations:

(1) $a^2 - bc = 5$
(2) $b^2 - ac = 7$
(3) $c^2 - ab = 9$

First, multiply equation 1 by b, equation2 by c, and equation 3 by a:

$a^2b - b^2c = 5b$

$b^2c - c^2a = 7c$

$c^2a - a^2b = 9a$

Add all three equations above to get:

(4) $9a + 5b + 7c = 0$

Second, multiply equation 1 by c, equation2 by a, and equation 3 by b:

$a^2c - c^2b = 5c$

$b^2a - a^2c = 7a$

$c^2b - b^2a = 9b$

Add all three equations above to get:

(5) $7a + 9b + 5c = 0$

Next, let's express b and c in terms of a.

Next, let's express c in terms of a by canceling out the b term from equations (4) and (5). To do so, multiply equation (4) by 9 and equation (5) by -5:

$81a + 45b + 63c = 0$

$-35a - 45b - 25c = 0$

Then add the two equations:

$46a + 38c = 0$

(6) $c = (-46/38)a$

Next, let's express b in terms of a by canceling out the c term from equations (4) and (5). To do so, multiply equation (4) by -5 and equation (5) by 7:

$-45a - 25b - 35c = 0$

$49a + 63b + 35c = 0$

Then add the two equations:

$4a + 38b = 0$

(7) $b = (-4/38)a = (-2/19)a$

Let's go back to equation 1 and substitute equations (6) and (7) for the b and c terms:

$a^2 – bc = 5$

$a^2 – (-2/19)a \; * \; (-46/38)a = 5$

$a^2 – (92/722)\, a^2 = 5$

$(630/722)\, a^2 = 5$

$(315/361)\, a^2 = 5$

$a^2 = 5 * (361/315)$

$a^2 = 1805/315 = 361/63$

$$a = +/- \sqrt{\frac{361}{63}} =\sim +/- 2.393775$$

Let's go with the negative value for a of -2.393775.

From equation (7) we can solve for b as:

$$b = \frac{2}{19}\sqrt{\frac{361}{63}} =\sim 0.251976$$

From equation (6) we can solve for c as:

$$c = \frac{23}{19}\sqrt{\frac{361}{63}} =\sim 2.897728$$

If we had gone with the positive value for a, then b and c would both have a negative term.