

Question: A fair coin is flipped 40 times. What is the probability of seeing both a run of at least five heads and at least five tails?

Answer: $107,094,548,225 / 549,755,813,888 = \text{apx. } 19.4804\%$.

Solution:

This is a classic Markov Chain problem. Let's call a "run" a string of five of the same side in a row. After the first flip, there will be ten possible states the flipper can be in:

- 0,1: Zero runs completed. Last flip was first flip or alternated sides.
- 0,2: Zero runs completed. Last two flips the same, but not three.
- 0,3: Zero runs completed. Last three flips the same, but not four.
- 0,4: Zero runs completed. Last four flips the same, but not five.
- 0,5: Zero runs completed. Last five or more flips the same.
- 1,1: One run completed. Last flip was first flip or alternated sides.
- 1,2: One run completed. Last two flips the same, but not three.
- 1,3: One run completed. Last three flips the same, but not four.
- 1,4: One run completed. Last four flips the same, but not five.
- 2: Both runs completed

I describe each state with two numbers. The first is the number of unique runs completed. The second is the number of consecutive flips going of the same side.

Since the outcome of a flip has two possibilities only, there are two states one can change to from any given state. The only exception is when the goal is already accomplished, in which case further flipping is a waste of time.

The following table is called a transition matrix. It shows the state of the last flip along the left column and the next state along the top row. The body is the probability of going from the current state to the next state on the next flip. For example, if the last four flips were HTTT (state 0,3), and no runs have been completed yet, then the next flip could be a heads, bringing the player to a 0,1 state or tails, bring the flipper to a 0,4 state.

$M^7 = M^6 * M$ will be the transition matrix after 7 flips:

0.437500	0.226563	0.117188	0.062500	0.093750	0.039063	0.015625	0.007813	0.000000	0.000000
0.406250	0.210938	0.109375	0.054688	0.125000	0.054688	0.023438	0.007813	0.007813	0.000000
0.343750	0.179688	0.093750	0.046875	0.179688	0.085938	0.039063	0.015625	0.007813	0.007813
0.226563	0.117188	0.062500	0.031250	0.281250	0.140625	0.070313	0.031250	0.015625	0.023438
0.000000	0.000000	0.000000	0.000000	0.476563	0.242188	0.125000	0.062500	0.031250	0.062500
0.000000	0.000000	0.000000	0.000000	0.460938	0.234375	0.117188	0.062500	0.031250	0.093750
0.000000	0.000000	0.000000	0.000000	0.429688	0.218750	0.109375	0.054688	0.031250	0.156250
0.000000	0.000000	0.000000	0.000000	0.367188	0.187500	0.093750	0.046875	0.023438	0.281250
0.000000	0.000000	0.000000	0.000000	0.242188	0.125000	0.062500	0.031250	0.015625	0.523438
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	1.000000

$M^{39} = M^{32} * M^7$ will be the transition matrix after 39 flips.

0.134349	0.069699	0.036159	0.018759	0.282331	0.140987	0.070354	0.035080	0.017478	0.194804
0.124617	0.064650	0.033540	0.017400	0.282658	0.141344	0.070634	0.035274	0.017603	0.212281
0.105858	0.054918	0.028491	0.014781	0.282580	0.141670	0.070990	0.035553	0.017796	0.247362
0.069699	0.036159	0.018759	0.009732	0.281040	0.141593	0.071317	0.035910	0.018076	0.317715
0.000000	0.000000	0.000000	0.000000	0.275337	0.140053	0.071239	0.036237	0.018432	0.458703
0.000000	0.000000	0.000000	0.000000	0.265961	0.135284	0.068813	0.035003	0.017804	0.477135
0.000000	0.000000	0.000000	0.000000	0.247529	0.125908	0.064044	0.032577	0.016571	0.513371
0.000000	0.000000	0.000000	0.000000	0.211292	0.107476	0.054669	0.027808	0.014145	0.584611
0.000000	0.000000	0.000000	0.000000	0.140053	0.071239	0.036237	0.018432	0.009376	0.724663
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	1.000000

As stated before, after the first flip the player will be in state 0,1. The top row of M^{39} shows the probability of being in all 10 states starting from state 0,1 after 39 more flips. The right column represents the (2) state. So, the upper right corner of the matrix is our answer, which rounds to 0.194804.

If we multiply that value in the upper right corner of M^{39} , by 2^{39} , the possible combinations of 39 more flips, we get 214,189,096,450. There are $2^{40} = 1,099,511,627,776$ possible ways to flip a coin 40 times. Thus, the answer is $214,189,096,450 / 1,099,511,627,776$, which reduces to $107,094,548,225 / 549,755,813,888 = \text{apx. } 19.4804\%$.

One can easily do all this matrix algebra in Excel using the MMULT function.