Question: The Michigan Lottery has a game with the following rules:

1. Three players.
2. There are 100 cards, numbered 1 to 100.
3. Each player will pick a card.
4. After looking at his card, the player may keep that card or put it back and take a new card.
5. The player with the highest card at the end shall win.
6. All cards are chosen with replacement. In other words, after a card is chosen it will be randomly placed back in the deck.
7. The players acting second and third have no knowledge of how the previous players did.

Assume that all three players are perfect logicians and there is no communication, including by tells, between the players. What is the optimal strategy?

Answer

To answer this question I will first answer a different question of what would be the strategy if instead each player received a random number from 0 to 1 distributed uniformly. Once I have that answer I will then simply extend it to a discrete distribution from 1 to 100.

It stands to reason that the optimal strategy is to stand at some number or higher and switch below it. The question is finding that point. On a continuous distribution there must be some number at which the player is indifferent between standing and switching. The problem is to find that number. Let's call is x.

If the player's first number is exactly x then his probability will be the same standing and switching. If he stands then the only way he can win is if both other players got less than x on both their first and second cards. The probability of this is x^4.

It gets more complicated calculating the probability of winning if the player switches. Here a random number must beat the better of number of the other two players. There are three possible situations that could have happened with the other two players:

- Situation A: Both player got less than *x* on their first tries.
- Situation B: One player got less than *x* on his first try and the other got more.
- Situation C: Both players got more than *x* on their first tries.

Let's look at each situation one at a time.

Situation A

The probability of both other players getting under x on their first tries is x^2. The probability that a random number t (of the third player) will beat two random numbers of the other two players, is t^2.

To summarize situation A:

Probability if situation A = x^2.
Probability distribution function = t^2.

Situation B

In this situation one competitor stood and one switched. So, the third player is competing against the better of a random number from 0 to 1 and another from x to 1. The third player's second number must be at least x to have any hope, because one of the other players already stood on a number greater than x.

The probability of situation this situation is $2x(1-x)$.

The probability that the third player's random number, let's call it t again, beats the switching player's number is t.

The probability that the third player's random number beats the standing player's number is (t-x)/(1-x), because that number must be somewhere between x and 1.

Thus, the probability the third player's random number beats both other players is t×(t-x)/(1-x) = $(1/(1-x))\times(t^2-tx)$.

To summarize situation B:

Probability if situation B = $2x(1-x)$.
Probability distribution function = $(1/(1-x))\times(t^2-tx)$.

Situation C

In this situation both competitors stood. So, the third player is competing against the better of two numbers from x to 1. The third player's second number must be at least x to have any hope.

The probability of situation this situation is $(1-x)^2$.

The probability that the third player's random number, let's call it t again, beats two other numbers distributed from x to 1 is

$[(t-x)/(1-x)]^2$ =

$1/(1-x)^2 \times (t^2 - 2tx + x^2)$, assuming the second number is at least x.

To summarize situation C:

Probability if situation C = $(1-x)^2$.
Probability distribution function = $[(t-x)/(1-x)]^2$.

Now we can determine the overall probability of a random number beating two other players. It is the dot product of the probability of the three situation situations and the probability of winning under each of them, which is:

$$x^2 \int_0^1 t^2\, dt \;+\; 2x(1-x) \int_x^1 \left(\frac{1}{1-x}\right) \times (t^2 - tx)\, dt \;+\; (1-x)^2 \int_x^1 \left(\frac{1}{1-x}\right)^2 \times (t^2 - 2tx + x^2)\, dt$$

$= x^2 \times (t^3/3)$ from 1 to 0
$+ 2x(1-x)/(1-x) \times [t^3/3 - t^2x/2$ from 1 to x]
$+ (1-x)^2/(1-x)^2 \times [t^3/3 - t^2x + x^2$ from 1 to x]

$= x^2 / 3$
$+ 2x \times [\, 1/3 - x/2 - x^3/3 + x^3/2\,]$
$+ 1/3 - x + x^2 - [\, x^3/3 - x^3 + x^3]$

$x^2 / 3 + x^4/3 - x^2 + 2x/3 - x^3/3 + x^2 - x + 1/3$

Remember that this equals the probability of winning by standing on x, which we've shown is x^4.

So $x^2 / 3 + x^4/3 - x^2 + 2x/3 - x^3/3 + x^2 - x + 1/3 = x^4$

$x^4 / 3 - x^3/3 + x^2/3 - x/3 + 1/3 = x^4$

$x^4 - x^3 + x^2 - x + 1 = 3x^4$

$2x^4 + x^3 - x^2 + x - 1 = 0$

Forgive me if I don't give you an exact expression for x, but it is approximately 0.69115.

If you extend this to a discrete distribution from 1 to 100, this would fall in the 70th card. So, with less than 70 the player should switch and with more he should stand. With exactly 70 we look at the how that 70th card would be divided. 11.5% would favor switching and 88.5% would favor standing. Since most of that 70th card favors standing, we stand. Thus, the optimal strategy is to switch on 69 or less and stand on 70 or more.