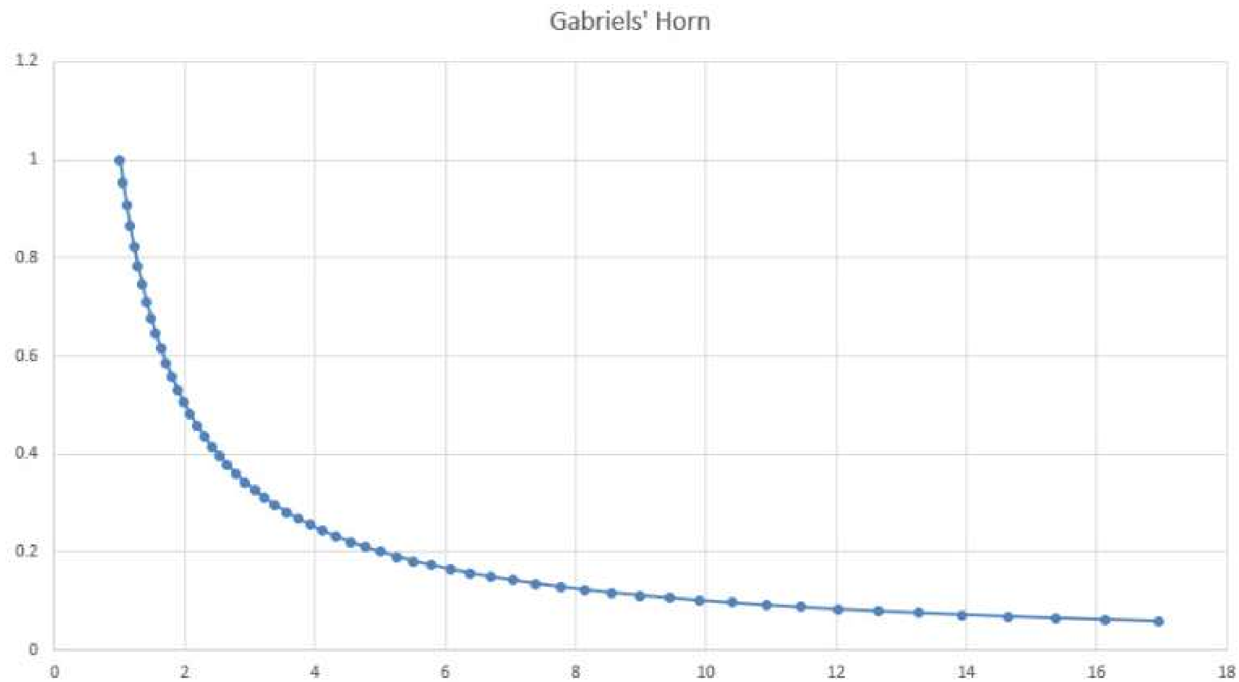
Consider the graph of y=1/x, for values of x from 1 to infinity. It will look like this for the values of x from 1 to about 17.



Then, rotate that graph around the x axis. It will make an infinitely long horn. The end of it will look like one of these horns you sometimes see at sporting events. It is known as Gabriel's Horn. The following image shows Gabriel blowing the end section of one.

Image source: soulofmathematics.com

Volume of Gabriel's Horn.

$\int_1^\infty \pi(\frac{1}{x})^2 dx$ =

$$\pi \int_1^\infty x^{-2} dx =$$

- π ($\frac{1}{x}$ for x from 1 to ∞) =

π

Surface Area of Gabriel's Horn.

y = $\frac{1}{x}$

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

Surface area = $\int_1^\infty \pi\ \frac{2}{x}\ \sqrt{1 + (\frac{dy}{dx})^2} dx$ =

$2\pi \int_1^\infty \frac{1}{x}\ \sqrt{1 + (-\frac{1}{x^2})^2} dx$ =

$2\pi \int_1^\infty \frac{1}{x}\ \sqrt{1 + \frac{1}{x^4}}\ dx$ =

Let's put aside the $\sqrt{1 + \frac{1}{x^4}}$ term for the moment, noting the expression will always be greater than 1 for any 1<= x <=∞.

$2\pi \int_1^\infty \frac{1}{x}\ dx$ =

2π (ln(x) for x = 1 to ∞) =

2π (ln(∞) – ln(1)) =

2π (∞ – 0) =

∞

If this equals ∞, then adding $\sqrt{1 + \frac{1}{x^4}}$ to the integral, which, again, is always equal or greater to 1, will still equal ∞.

This leads to the question of how can the volume be π and the surface area infinite? It is known as Gabriel's Horn paradox. There is a lot of content about it on the Internet. My simple explanation is the perceived paradox comes from comparing two-dimensions with three. That just isn't kosher mathematically.

However, to take it a little deeper, consider instead of a smooth horn we have a sequence of cans. Each can will have a radius of $\frac{1}{i}$, where i is an infinite series from 1 to infinity, and a height of 1.

The volume of this rough approximation of Gabriel's Horn is:

$$\sum_{i=0}^{\infty} \pi \ \frac{1}{i^2} =$$

$$\pi \sum_{i=0}^{\infty} \ \frac{1}{i^2} =$$

$$\pi \times \ \frac{\pi^2}{6} =$$

$$\frac{\pi^3}{6} =\sim 5.167713.$$

The infinite series of $\sum_{i=0}^{\infty} \frac{1}{i^2}$ is a well-known to sum of $\frac{\pi^2}{6}$.

We could get a better estimate of the volume by slating the edges, to make truncated cones. However, I above overestimate at least argues that the volume of something bigger than Gabriel's Horn is finite.

The surface area of just the sides of the cans is:

$$\sum_{i=0}^{\infty} 2\,\pi\ \frac{1}{i} =$$

$$2\,\pi \sum_{i=0}^{\infty} \frac{1}{i} =$$

$$2\,\pi \times \infty = \infty$$

This infinite series of $\sum_{i=0}^{\infty} \frac{1}{i}$ is known as the harmonic series and sums to infinity.

What I hope I showed with these approximations is that the volume of Gabriel's Horn is finite, while the surface area is infinite.