Question:

Imagine there are two busses, as follows:

- Bus A arrives at the bus stop exactly once an hour.
- Bus B arrives at a random time, uniformly distributed, every hour, with the range starting and ending on the hour (for example 3:00 PM to 4:00 PM).

Assuming no passengers own a watch and simply choose a bus and then wait for the next one to arrive. Your questions are as follows:

1. What is the average time between arrivals for bus A?
2. What is the average time between arrivals for bus B?
3. What is the waiting time for bus A?
4. What is the waiting time for bus B?

Answer:

1. 60 minutes
2. 60 minutes
3. 30 minutes
4. 35 minutes

Solution:

The answers to 1, 2 and 3 are obvious.

Following is my solution to #4.

Let x be the time a random bus arrives before the end of the hour, where x is measured in hours.

Let y be the time a random bus arrives after the start of the hour, where x is measured in hours.

The time between buses x and y is (x+y). If we assume one passenger per hour, the probability this passenger will get on bus y is (x+y). The average waiting time is (x+y)/2. Thus, the expected waiting time is $(x+y)^2$ / 2.

We need to integrate over x and y to find the average waiting time.

$$\left(\frac{1}{2}\right) \times \int_0^1 \int_0^1 (x+y)^2 dx\, dy =$$

$$\left(\frac{1}{2}\right) \times \int_0^1 \int_0^1 x^2 + 2xy + y^2 dx\, dy =$$

$$\left(\frac{1}{2}\right) \times \int_0^1 \frac{x^3}{3} + x^2 y + y^2\ (for\ x = 0\ to\ 1) dy =$$

$$\left(\frac{1}{2}\right) \times \int_0^1 \frac{1}{3} + y + y^2\, dy =$$

$$\left(\frac{1}{2}\right) \times \left(\frac{y}{3} + \frac{y^2}{2} + \frac{y^3}{3}\right) for\ y = 0\ to\ 1 =$$

$$\left(\frac{1}{2}\right) \times \left(\frac{1}{3} + \frac{1}{2} + \frac{1}{3}\right) = \frac{7}{12}$$

What is interesting is the expected time between buses is the same for A and B, but the expected waiting time for the passenger is more with bus B. In simple English, when there is a long gap between buses, more passengers will be waiting. In other words, a random passenger is more likely to show up during a long time between buses.