Question

$1^3 + 2^3 + 3^3 + \ldots + n^3 =$

Answer

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \sum_{i=1}^{n} i^3 = \frac{n^2\,(n^2+1)}{4}$$

Solution

We have established from earlier problems that:

$1+2+3+\ldots+n = \frac{n(n+1)}{2}$

$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n\,(2n+1)\,(n+1)}{6}$

Now we're ready to move onto $1^3 + 2^3 + 3^3 + \ldots + n^3$

The humdinger with this solution is to use telescoping sums. Note that:

$x^4 - (x - 1)^4 =$

$x^4 - (x^4 - 4x^3 + 6x^2 - 4x + 1) =$

$4x^3 - 6x^2 + 4x - 1$

Next consider:

$x^4 - (x - 1)^4 = 4x^3 - 6x^2 + 4x - 1$

$(x - 1)^4 - (x - 2)^4 = 4(x-1)^3 - 6(x - 1)^2 + 4(x - 1) - 1$

$(x - 2)^4 - (x - 3)^4 = 4(x-2)^3 - 6(x - 2)^2 + 4(x - 2) - 1$

$(x - 3)^4 - (x - 4)^4 = 4(x-3)^3 - 6(x - 3)^2 + 4(x - 3) - 1$

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$(2)^4 - (1)^4 = 4(2)^3 - 6(2)^2 + 4(2)^1 - 1$

$(1)^4 - (0)^4 = 4(1)^3 - 6(1)^2 + 4(1)^1 - 1$

Next, add up the x terms. Note all but two cancel out on the left side:

x^4 - 0^4 = 4$\sum_{i=1}^{x} i^3$ - 6$\sum_{i=1}^{x} i^2$ + 4$\sum_{i=1}^{x} i$ - $\sum_{i=1}^{x} 1$

Let's isolate $\sum_{i=1}^{x} i^3$ on the left side, since that what's we're trying to solve for.

4$\sum_{i=1}^{x} i^3$ = x^4 + 6$\sum_{i=1}^{x} i^2$ - 4$\sum_{i=1}^{x} i$ + $\sum_{i=1}^{x} 1$

Recall, from the beginning that:

$\sum_{i=1}^{n} i =$ 1+2+3+…+n = $\frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2 =$ $1^2 + 2^2 + 3^2 + \ldots + n^2$ = $\frac{n\,(2n+1)\,(n+1)}{6}$

So, we have:

4$\sum_{i=1}^{n} i^3$ = n^4 + 6$\left(\frac{n\,(2n+1)(n+1)}{6}\right)$ – 4$\left(\frac{n(n+1)}{2}\right)$ + n

= n^4 + $n\,(2n + 1)(n + 1)$ – $2n(n + 1) + n$

= n^4 + $2n^3 + 3n^2$ + n^1- $2n^3 - 2n^1$ + n^1

= $n^4 + n^2$

= $n^2\ (\ n^2$ + 1)

Dividing both sides by 4, we have:

$1^3 + 2^3 + 3^3 + \ldots + n^3$ = $\sum_{i=1}^{n} i^3$ = $\frac{n^2\ (\ n^2+1)}{4}$