Problem: A player plays video poker. The probability of hitting a royal flush on any hand is $p$. What is the probability that in $n$ hands there will be no “drought” of $d$ consecutive hands without a royal flush?

Solution 1. Assume that the player plays at a constant speed, and choose the unit of time such that expected number of royal flushes per unit time is 1.

Let $X_i$ be the time of the $i$th royal flush, we can model this with the jump times of a homogeneous Poisson process with parameter 1. We can consider $X_i$ ($i = 1, 2, 3, \ldots$) as a sequence of random variables such that $X_1$, $X_2 - X_1$, $X_3 - X_2$, etc., are independent exponentially distributed random variables with parameter 1.

The question can be restated in these terms as follows. Given $k$ and $x$, what is the probability that $X_1$, $X_2 - X_1$, $\ldots$, $X_m - X_{m-1}$ and $x - X_m$ are all at most $k$, where $X_m$ is the largest $m$ such that $X_m < x$? (In the original question, $d = 200000$, $n = 1000000$, $p = 1/40391$, so $k = 200000/40391$, $x = 1000000/40391$.)

Let this probability be $f(x)$. Clearly $f(x) = 1$ if $0 \leq x \leq k$. If $x > k$, then consider $X_1$. We must have $X_1 \leq k$, and then $X_i - X_{i-1}$ ($i = 2, 3, \ldots, m$) and $x - X_m$ must all be at most $k$, which has probability $f(x - X_1)$. This gives the following equation for $f(x)$ if $x > k$:

$$f(x) = \int_0^k e^{-t}f(x - t) \, dt. \quad (1)$$

By using the Dirac $\delta$-function, this can be rewritten in a form valid for all $x$:

$$f(x) = \int_0^k e^{-t}(f(x - t) + \delta(x - t)) \, dt. \quad (2)$$

(The $\delta$-function is not a “real” function, it is a so-called generalized function or distribution. It can be considered as a notational convenience, it is defined by the property that $\int_{-\infty}^{\infty} g(x)\delta(x) \, dx = g(0)$ for any function $g$. In the above equation it adds $e^{-x}$ to the right-hand side if $0 \leq x \leq k$ to make it correct for all $x > 0$.)

The Laplace transform $\mathcal{L}(g)$ of a function $g(t)$ defined on $[0, \infty)$ is

$$\mathcal{L}(g)(s) = \int_0^{\infty} g(t)e^{-st} \, dt.$$
Let \( h(x) = e^{-x} \) if \( 0 \leq x \leq k \), and 0 otherwise. Then the right-hand side of (2) is the convolution of \( h(x) \) with \( f(x) + \delta(x) \), and it is a well-known property of Laplace transforms that the Laplace transform of the convolution is the product of the Laplace transforms of the factors. Hence we obtain

\[
\mathcal{L}(f) = \mathcal{L}(h)(\mathcal{L}(f) + \mathcal{L}(\delta)).
\]

\( \mathcal{L}(\delta) = 1 \) and \( \mathcal{L}(h) = \frac{1 - e^{-k(1+s)}}{1 + s} \). We can solve for \( \mathcal{L}(f) \),

\[
\mathcal{L}(f) = \frac{\mathcal{L}(h)}{1 - \mathcal{L}(h)} = -1 + e^{k(1+s)} \frac{1 + e^{k(1+s)}}{1 + e^{k(1+s)s}}.
\]

(3)

The middle expression can be expanded into a power series, so \( \mathcal{L}(f) = \sum_{r=1}^{\infty} [\mathcal{L}(h)]^r \). The Laplace transform converts convolutions to products, therefore by inverting it we get \( f(x) = \sum_{r=1}^{\infty} \ast^r h(x) \), where \( \ast^r h(x) \) is the \( r \)-fold convolution of \( h \) with itself. This seems to be the most explicit form of \( f(x) \) but it does not appear to be useful for calculations.

Instead of an exact form we need to find an approximation. The asymptotic behaviour of \( f(x) \) as \( x \to \infty \) is determined by the poles of \( \mathcal{L}(f) \), the places where \( \mathcal{L}(f) \) is not defined because the denominator is 0. Both real and complex values need to be considered. Near a real pole \( s = a \), \( \mathcal{L}(f) \) behaves like \( A/(s - a) \) for some constant \( A \), and this corresponds to a term \( Ae^{ax} \) in \( f(x) \). Complex poles come in conjugate pairs, \( b \pm ci \), and their joint effect is a term of the form \( e^{bx}(\alpha \cos(cx) + \beta \sin(cx)) \).

We need to find where the denominator of \( \mathcal{L}(f) \) vanishes. \( 1 + e^{k(1+s)s} = 0 \) cannot be solved for \( s \) using standard functions. It can be rearranged to the form

\[
kse^{ks} = -ke^{-k}.
\]

(4)

\( s = -1 \) is clearly a solution, but there is no pole there unless \( k = 1 \), because the numerator of \( \mathcal{L}(f) \) also vanishes. There is a function called productlog, sometimes denoted by \( W \), which is the inverse of \( ze^z \), so that it satisfies \( z = W(z)e^{W(z)} \). Let

\[
a = \frac{W(-ke^{-k})}{k},
\]

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where we are using the non-standard convention that out of the two possible real values of $W$ we take the one not equal to $-k$, unless $k = 1$, in which case the only possible value is $-1$. $s = a$ and $s = -1$ are the only real solutions of $1 + e^{k(1+s)s} = 0$ and $\mathcal{L}(f)$ has a pole at $s = a$.

If $k = 1$ then $a = -1$, if $k \neq 1$, then $k$ can be expressed in terms of $a$ as

$$k = \frac{-\ln(-a)}{1 + a}.$$ 

For practical purposes $a$ can be calculated by solving this equation numerically to the required accuracy.

We now claim that $f(x)$ is asymptotically equal to $Ae^{ax}$ for some $A$. The only real pole of $\mathcal{L}(f)$ is at $s = a$, if the dominant term of $f(x)$ came from a pair of complex poles, then it would be of the form $e^{bx}(\alpha \cos(cx) + \beta \sin(cx))$, but since it oscillates and changes sign, it cannot be the dominant term in $f(x)$ which is positive and monotonically decreasing. (This might also follow from certain properties of the complex values of the $W$ function, which may be known to people who deal with it more often than I do.)

The coefficient $A$ can be determined as

$$A = \lim_{s \to a} (s - a)\mathcal{L}(f)(s) = \lim_{s \to a} \frac{(s - a)(-1 + e^{k(1+s)s})}{1 + e^{k(1+s)s}}.$$

If $k \neq 1$, L'Hôpital’s rule gives

$$A = \frac{(1 + a)^2}{1 + a - a \ln(-a)} = \frac{1 + a}{1 + ak}.$$ 

while if $k = 1$, L'Hôpital’s rule needs to be applied twice to get $A = 2$. Therefore if $k \neq 1$,

$$f(x) \approx \frac{1 + a}{1 + ak}e^{ax},$$

while if $k = 1$,

$$f(x) \approx 2e^{-x}.$$ 

Summary

Let $p$ be the probability of the royal flush, $d$ the length of the “drought”, $n$ the total number of hands played.
1. Set $k = dp$, $x = np$.
2. If $k = 1$ then let $a = -1$, otherwise find $a$ such that $k = -\ln(-a)/(1 + a)$. ($a$ is a negative number, if $k > 1$ then $-1 < a < 0$, if $k < 1$ then $a < -1$, and $a$ needs to be calculated to high accuracy.)
3. If $k = 1$, then let $A = 2$, otherwise let $A = (1 + a)/(1 + ak)$.
4. The probability of no “drought” of length $d$ in $n$ hands is approximately $Ae^{ax}$.

In the original problem, $k = 200000/40391 = 4.9516$, $x = 1000000/40391 = 24.758$. Hence $a = -0.00733363$, $A = 1.03007$, and the probability of no “drought” is $f(x) \approx 0.859042$, and the probability of there being a “drought” is $1 - f(x) \approx 0.140958$.

Solution 2.

Let $b_n$ be the probability that in $n$ hands of video poker there is no royal flush “drought” of length $d$. $b_0 = b_1 = \ldots = b_{d-1} = 1$, and for $n \geq d$,

$$b_n = p \sum_{i=1}^{d} (1 - p)^{i-1} b_{n-i},$$

this is the discrete equivalent of (1).

Let

$$\phi(x) = x^d - p \sum_{i=1}^{d} (1 - p)^{i-1} d^{d-i} = \frac{x^{d+1} - x^d + p(1 - p)^d}{x - (1 - p)}.$$  

It can be verified that $\phi(x)$ has no multiple roots, so the exact formula is of the form $b_n = \sum_{i=1}^{d} \alpha_i x_i^n$, where the $x_i$ ($i = 1, 2, \ldots, d$) are the roots of the characteristic equation $\phi(x) = 0$.

The coefficients can be determined by using Joshua Green’s idea from [http://www.princeton.edu/~jvgreen/RandomEvent.pdf](http://www.princeton.edu/~jvgreen/RandomEvent.pdf) as

$$\alpha_i = \frac{(1 - p)^d}{\phi'(x_i)(1 - x_i)}.$$  

$\phi(x)$ has a root of maximum modulus close to but slightly less than 1, call it $x_1$. For large $n$, all the terms in $b_n$ apart from $\alpha_1 x_1^n$ are negligible, and

$$b_n \approx \alpha_1 x_1^n = \frac{(1 - p)^d x_1^n}{\phi'(x_1)(1 - x_1)}.$$  

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Using the numbers in the original question, \( x_1 = 0.9999998184456574 \) and \( \alpha_1 = 1.03007 \), hence \( b_{1000000} = 1.03007 \times 0.9999998184456574^{1000000} = 0.859050 \) is the probability of no “drought” of length 200000 in 1000000 hands of video poker, in excellent agreement with the previous solution.

While this second method may give a more accurate result in theory, calculating \( x_1 \) to sufficient accuracy is not simple.