Loss rebates

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1 Introduction

The game is defined by a list of payouts u_1, u_2,..., u_ℓ, and a list of probabilities p_1, p_2, ..., p_ℓ, $\sum_{i=1}^{\ell} p_i = 1$. We allow u_i to be rational numbers, not just integers, to include games like blackjack, n-play video poker or the banker bet in baccarat. We assume that the casino has the advantage, so $\sum_{i=1}^{\ell} p_i u_i < 0$.

The player can bet any positive integer k up to his current bankroll or the maximum bet b, whichever is smaller, and his bankroll increases by $u_i k$ with probability p_i. To cater for blackjack or poker type games, only the player's initial bet is limited, and we allow the player to borrow money for splits, doubles or raises if necessary, but he has to stop if he loses and ends up with a negative bankroll. Each game is independent of all others.

Several casinos have promotions where they give the player 10% of his losses each month or on certain days, and VIP Casino also gives a 1% bonus on the winnings. In this paper we investigate the value of these bonuses and the optimal strategy for them. We assume that the bonus is withdrawable without further wagering requirements.

2 The questions

If the player's initial bankroll is m, his current bankroll is n, and the casino offers a 10% rebate on losses, what strategy should the player follow to maximize his expectation and what is this expectation a_n? What if the casino also offers a 1% bonus on winnings?

n can be a rational number whose denominator is the least common multiple of the denominators of the u_i, for example, when dealing with blackjack, n can be a half integer, but the stake is always an integer and let us also define $a_n = n + 0.1m$ for $n < 0$.

Define the *value of the promotion* to be the amount the player expects to gain by playing optimally instead of cashing in immediately.

In games involving an element of skill, we assume that the player is playing a fixed strategy, possible adjustments to playing strategy in view of the changed expected values are not considered, strategy will only mean betting strategy.

3 The solution

These problem can be solved by the same iterative method as the phantom bonus problem, so the details are not repeated here. For the simple 10% rebate the starting values need to be set to $a_{n,0} = 0.9n+0.1m$ for $0 \le n \le m$, and $a_{n,0} = n$ for $n > m$. For the VIP Casino problem, set $a_{n,0} = 1.01n - 0.01m$ for $n > m$. There are *a priori* bounds on how high the player should aim, he should stop if his bankroll exceeds $\left(m \sum_{u_i<0} p_i u_i\right) \Big/ \left(10 \sum_{i=1}^{\ell} p_i u_i\right)$ in the case of the loss rebate, or $\left(0.11m \sum_{u_i<0} p_i u_i\right) \Big/ \left(1.01 \sum_{i=1}^{\ell} p_i u_i\right)$ in the case of the VIP Casino promotion.

4 The results

The tables in the Appendix show the results for various games. The player's initial bankroll was always taken to be $m = 100$, this way the maximum amount of the loss rebate is 10 and the figures can be compared to those obtained for the phantom bonus. The probabilities for blackjack were taken from a simulation by Michael Shackleford, the "Wizard of Odds", available on the web at http://www.wizardofodds.com/games/blackjack/bjapx4.html.

The first table for each game is for the 10% loss rebate, the second table for the VIP Casino promotion. In each table the first column shows the maximum bet, the second and third columns the smallest and largest bankrolls, respectively, at which it is still in the player's interest to play, rather than to cash in. If the number in the second column is 1, it means that the player should play until his bankroll exceeds the number in the third column or go bust. The fourth column shows the value of the promotion at the initial bankroll.

These calculations required much more computer time than for the phantom bonus, the most difficult case was Cryptologic Double Bonus video poker with 1% bonus on winnings with $b = 1$, this took about 16 hours on a computer with a 2.2 GHz Intel Pentium processor. This is probably due to the fractions involved. The calculations had to be done using exact arithmetic, the standard 16 digits numerical precision was not enough, it sometimes caused the iterations to diverge. Some calculations were done using smaller numbers (e.g., $m = b = 10$ instead of $m = b = 100$) and then scaled up. As a consequence, the numbers in the second and third columns are not always exact, but they are always accurate within one maximum bet.

There are some new phenomena with the loss rebate which do not occur with the phantom bonus. It is clearly never in the player's interest to stop if his bankroll does not exceed the phantom bonus. With the loss rebate, if the house edge is too high, the player should simply not play. Such games are the tie bet in baccarat and the coin flip with the probability of winning 0.45, these games are not included in the tables. If it is not in the player's interest to play at the initial bankroll, he should not play at all. Even for other games, the strategy is not always "aim for a high target or go bust", but "aim for a high target or stop at a stop loss limit greater than 0". For example, when playing blackjack with the 10% loss rebate, the player should stop when his losses exceed 7.5 maximum bets or when his winnings exceed 7 maximum bets, with the additional 1% bonus on profits, the limits are 8 maximum bets either way. This assumes that the initial bankroll is more than 7.5 or 8 maximum bets. The additional 1% bonus on winnings calls for a slightly more aggressive strategy in general.

It is interesting to note that when the stop loss limit it 0, the target bankroll for the 10% loss rebate on 100 units is the same as for the 10 unit phantom bonus, the calculations in the next section will explain this. This also shows that the loss rebate requires a much less aggressive strategy.

Many observations are the same as for the phantom bonus. The value of the promotions, and also the upper bound on the bankroll at which the player should stop playing both increase with b, while the lower bound decreases with b. The value of these promotions is maximal at the initial bankroll.

The house edge is not the primary factor in determining the value of the promotions, a large probability of losing and a small probability of winning a large amount is the good for the player. Among all the roulette bets, which all have the same house edge, betting on a single number is the best. Similarly, Jacks or Better video poker with doubling is better than without doubling, while the house edge is the same. The line bet (6 numbers) in roulette with house edge 2.7% is better for the player than coin flip with probability of winning 0.495 and house edge 1%. The numbers for coin flip with probability of winning 0.499 and house edge 0.2% are similar to those for Jacks or Better video poker with house edge 0.46%. The best game is Cryptologic Double Bonus video poker, which combines both a low house edge (0.06%) and the possibility of big wins. The value of the VIP Casino promotion for this game with large enough bets may exceed 10, of course, this comes at the expense of extremely large variance.

The value of the loss rebate is usually less than that of the phantom bonus, it is possible to realize about 90% of the nominal value of the phantom bonus on most games with high enough maximum bet, while only video pokers, the single number bet in roulette and the coin flip with the probability of winning 0.499 achieve more than 70% with the loss rebate. The extra 1% bonus on winnings increases the value of the promotion by about 15%.

5 Details of the strategy

Let kb be the largest and $k'b$ the smallest multiple of the maximum bet at which the player should still play. They can be calculated by a similar method as in the case of phantom bonuses.

Assume that all the payouts are integers ≥ -1. Empirical evidence suggests that if player's bankroll is a multiple of b, he should always bet the maximum, b. Let $u = \max\{u_i | 1 \leq i \leq \ell\}$. The numbers a_{jb} ($k' \leq j \leq k$) satisfy the recurrence relation $a_{jb} = \sum_{i=1}^{\ell} p_i a_{(j+u_i)b}$ together with the boundary conditions $a_{(k'-1)b} = 0.9(k'-1)b + 0.1m$ and $a_{(k+1)b} = (k+1)b$, $a_{(k+2)b} = (k+2)b, \ldots$, $a_{(k+u)b} = (k+u)b$ in the case of the 10% loss rebate. In the case of the VIP Casino promotion, the equations have to be changed to $a_{(k+1)b} = 1.01(k+1)b - 0.01m$, $a_{(k+2)b} = 1.01(k+2)b - 0.01m, \ldots$, $a_{(k+u)b} = 1.01(k+u)b - 0.01m$.

Given a pair of positive integers $t' \leq m/b$ and $t \geq m/b$, we can define

$\alpha_{jb,t',t}$ by substituting it for a_{jb} in the above equations. The pair is *suitable* if $a_{t'b,t',t} \geq 0.9t'b + 0.1n$ and $a_{tb,t',t} \geq tb$ in the case of the 10% loss rebate, while in the case of the VIP Casino promotion, the second inequality has to be replaced by $a_{tb,t',t} \geq 1.01tb - 0.01m$.

k' is the smallest possible value of t' and k is the largest possible value of t in a suitable pair, they can be found by searching, but there does not appear to be any method of finding them directly.

If $k' = 1$, meaning that the player should play until he either reaches his goal or goes bust, the 10% loss rebate with initial bankroll 100 units and the phantom bonus of 10 units give the same equations for k, this explains the observation in the previous section.

If the maximum bet is large enough or there is no maximum bet and all the payouts are integers greater than or equal to -1, then the player should bet his whole bankroll until he either reaches the goal given in Section 3 or goes bust.

It might be expected that the player should always bet the table limit or his whole bankroll, and just like in the case of the phantom bonus, this is usually correct, but not always. With the phantom bonus, the correct strategy was always to bet either the maximum or difference between the current bankroll and the largest multiple of the maximum bet not exceeding it, so that if the player lost, his bankroll would be an integer multiple of the maximum bet. There is a tendency to favor multiples of the maximum bet with the loss rebate, too, but the bet size may be such that the player's bankroll will be equal to or just slightly greater than an integer multiple of the maximum bet if he wins. For even money bets, 1/2, 1/4 or 1/8 of the maximum bet may also be a desirable goal. The tables below show the bankroll and optimal bet size for various games to illustrate these principles.

Roulette, even money bets, $b = 100$, 10% loss rebate

n	52	53	54	55	...	70	71	72	73	74	75
bet	48	47	46	45	...	30	29	28	27	26	75

For $52 \leq n \leq 74$, the player aims to get to 100 if he wins.

Roulette, column or dozen bets, payout 2:1, $b = 100$, 10% loss rebate

n	21	22	23	24	...	33	34	35	36	37	38	39	40	41	42
bet	6	5	5	24	...	33	33	33	32	32	31	31	30	30	29

For $34 \leq n \leq 42$, the player aims to get to 100 or 101 if he wins.

Coin flip, probability of winning 0.49, $b = 100$, 10% loss rebate

n	34	35	36	37	38	39	40
bet	16	15	14	13	12	39	40

Coin flip, probability of winning 0.49, $b = 100$, VIP Casino promotion

n	27	28	29	30	31	32	33	34
bet	23	22	21	20	19	18	17	34

In the above two cases, for $34 \leq n \leq 38$ and for $27 \leq n \leq 33$, respectively, the player aims to get $50 = b/2$ if he wins.

Baccarat, player bet, $b = 100$, VIP Casino promotion

n	9	10	11	12	13	14	15	16	17
bet	4	10	11	12	12	11	10	9	17

For $13 \leq n \leq 16$, the player aims to get $25 = b/4$ if he wins. There is a similar example with $b = 200$, the optimal bet with a bankroll of 13 is 12.

The last two tables show cases where the optimal strategy cannot be explained easily.

Roulette, split bet (2 numbers), payout 18:1, $b = 50$, VIP Casino promotion

n	50	51	52	53	...	60	61	62	63	64	65	66	67	68	69
bet	50	1	50	50	...	50	50	1	50	50	50	50	17	18	19

Roulette, corner bet (4 numbers), payout 9:1, $b = 100$, VIP Casino promotion

n	101	102	103	104	105	...	109	110	111	112	113
bet	100	2	3	4	100	...	100	2	3	100	100

6 The effects of using the wrong strategy

As the previous section shows, the optimal betting strategy is quite far from obvious. Fortunately, the simple strategy of betting the whole bankroll or the maximum allowed, whichever is smaller, is quite good, and in the few cases I looked at it never caused a loss of more than 0.5% of the initial bankroll compared to the optimal strategy.

The other element of the strategy is the correct target and stop loss limit. Too timid a strategy will fail to realize the value of the loss rebate, while too agressive a strategy will also cause the player to lose because of the house edge. The correct target is often around three times the initial bankroll, if the player choose to double or to quadraple his bankroll instead, he may give up 0.5% of his initial bankroll, but this can mean 10–15% of the value of the loss rebate. Too agressive a strategy is more dangerous, playing on after reaching the correct target instead of cashing in can cost 5% or more of the initial bankroll and even wipe out any expected profit from the loss rebate.

7 Appendix

Baccarat (8 decks): player bet

1	99	101	0.048141
2	98	102	0.0962819
5	93	107	0.240705
10	86	114	0.48141
20	71	128	0.962819
50	28	170	2.40705
100	12	225	4.0399
200	8	286	4.85648
500	7	371	4.85648

1	99	101	0.057868
2	97	103	0.115736
5	92	107	0.28934
10	84	115	0.57868
20	67	131	1.15736
50	22	178	2.8934
100	9	236	4.68845
200	6	303	5.58615
500	5	404	5.60089

Blackjack

1	92.5	107	0.201784
2	85	114	0.403568
5	62.5	135	1.00892
10	25	170	2.01595
20	1	235	3.54223
50	1	350	5.35131
100	1	460	6.35234

1	92	108	0.242936
2	84	116	0.485872
5	60	140	1.21468
10	20	180	2.42494
20	1	245	4.14159
50	1	370	6.11084
100	1	500	7.25468

Coin flip, probability of winning 0.48

1	100	100	0.012
2	100	100	0.024
5	99	101	0.06
10	98	102	0.12
20	96	104	0.24
50	88	111	0.6
100	81	123	1.2

1	100	100	0.0168
2	100	100	0.0336
5	99	101	0.084
10	97	102	0.168
20	94	105	0.336
50	85	114	0.84
100	76	129	1.68

Coin flip, probability of winning 0.49

1	100	100	0.031	1	100	100	0.0359
2	99	101	0.062	2	99	101	0.0718
5	96	104	0.155	5	96	104	0.1795
10	92	108	0.31	10	91	109	0.359
20	83	116	0.62	20	81	118	0.718
50	56	141	1.55	50	51	146	1.795
100	34	179	3.1	100	27	190	3.59

Coin flip, probability of winning 0.495

1	98	102	0.0645228	1	98	102	0.0790729
2	96	104	0.129046	2	96	104	0.158146
5	90	110	0.322614	5	88	111	0.395365
10	79	121	0.645228	10	76	123	0.790729
20	57	142	1.29046	20	51	146	1.58146
50	11	203	3.13008	50	7	212	3.71214
100	4	270	4.73984	100	3	285	5.39322

Coin flip, probability of winning 0.499

1	88	112	0.328972	1	86	113	0.396093
2	75	124	0.657944	2	72	127	0.792186
5	36	162	1.64486	5	30	168	1.98047
10	1	222	3.17062	10	1	232	3.7247
20	1	309	4.75243	20	1	323	5.42949
50	1	478	6.4383	50	1	501	7.22636
100	1	660	7.39252	100	1	692	8.23941

European roulette: straight up (single number)

1	67	121	0.823681	1	64	124	0.991716
2	33	143	1.64736	2	27	149	1.98343
5	1	201	3.69786	5	1	212	4.28873
10	1	252	5.2068	10	1	268	5.94317
20	1	295	6.15924	20	1	317	7.00561
50	1	332	6.80058	50	1	358	7.7206
100	1	347	7.02703	100	1	376	7.97297

European roulette: split bet (2 numbers)

1	84	110	0.400049	1	83	112	0.481785
2	68	121	0.800097	2	65	124	0.96357
5	20	152	2.00024	5	13	160	2.40893
10	3	198	3.63066	10	1	209	4.21609
20	1	249	5.14822	20	1	265	5.88137
50	1	302	6.31848	50	1	324	7.187
100	1	327	6.75676	100	1	352	7.67568

European roulette: street bet (3 numbers)

1	90	106	0.258869	1	89	107	0.311842
2	80	113	0.517739	2	78	115	0.623684
5	50	133	1.29435	5	44	138	1.55921
10	9	167	2.5842	10	7	176	3.0839
20	2	216	4.25364	20	1	228	4.88587
50	1	277	5.85099	50	1	296	6.66947
100	1	309	6.48649	100	1	332	7.37838

European roulette: corner bet (4 numbers)

1	93	104	0.188205	1	92	105	0.226707
2	86	109	0.37641	2	84	111	0.453414
5	64	124	0.941024	5	60	127	1.13354
10	28	149	1.88205	10	20	155	2.26707
20	6	192	3.491	20	3	203	4.06435
50	2	256	5.3981	50	1	272	6.16801
100	1	293	6.21622	100	1	314	7.08108

European roulette: line bet (6 numbers)

1	96	103	0.11745	1	96	103	0.141873
2	92	106	0.234901	2	91	106	0.283746
5	79	115	0.587252	5	76	117	0.709364
10	57	130	1.1745	10	52	134	1.41873
20	18	160	2.34901	20	15	168	2.83746
50	5	221	4.53616	50	3	234	5.21329
100	3	263	5.67568	100	2	282	6.48649

European roulette: 9 numbers

1	98	101	0.0707495
2	96	103	0.141499
5	88	108	0.353748
10	76	117	0.707495
20	52	134	1.41499
50	14	183	3.35281
100	9	227	4.86486

1	98	102	0.084644
2	95	104	0.169288
5	87	110	0.42322
10	73	120	0.84644
20	46	140	1.69288
50	12	194	3.90175
100	6	240	5.59459

European roulette: dozen or column bets (12 numbers)

1	99	101	0.0470651
2	98	102	0.0941302
5	93	105	0.235325
10	86	111	0.470651
20	71	122	0.941302
50	36	156	2.35325
100	21	200	4.05405

1	99	101	0.0571817
2	97	102	0.114363
5	92	106	0.285908
10	84	112	0.571817
20	67	125	1.14363
50	30	164	2.85908
100	16	210	4.7027

European roulette: even money bets (18 numbers),
no en prison rule or half of your stake back on 0

1	100	100	0.0243243
2	100	100	0.0486486
5	98	102	0.121622
10	96	104	0.243243
20	91	109	0.486486
50	76	123	1.21622
100	52	147	2.43243

1	100	100	0.0291892
2	99	101	0.0583784
5	97	102	0.145946
10	94	105	0.291892
20	88	111	0.583784
50	70	128	1.45946
100	48	154	2.91892

Full pay Jacks or Better video poker

1	67	121	0.725652
2	34	143	1.4513
5	1	201	3.27731
10	1	264	4.6667
20	1	344	5.81142
50	1	480	6.98917
100	1	605	7.67061

1	62	124	0.908339
2	24	149	1.81668
5	1	212	3.88772
10	1	275	5.377
20	1	360	6.60106
50	1	505	7.83973
100	1	640	8.56562

Full pay Jacks or Better video poker with doubling once on every win

1	26	150	1.63572	1	14	157	2.04428
2	1	196	3.06156	2	1	207	3.65335
5	1	284	4.90873	5	1	299	5.63043
10	1	374	6.03806	10	1	390	6.83604
20	1	486	6.93285	20	1	510	7.78084
50	1	680	7.84278	50	1	720	8.74025
100	1	860	8.33571	100	1	910	9.2598

Cryptologic Double Bonus Video Poker

1	1	545	7.1477	1	1	574	8.00639
2	2	718	7.80724	2	2	756	8.70235
5	1	1025	8.46746	5	1	1080	9.39366
10	1	1335	8.8471	10	1	1410	9.79079
20	1	1710	9.13194	20	1	1810	10.0896
50	1	2280	9.39536	50	1	2420	10.3664
100	1	2740	9.52055	100	1	2920	10.5039